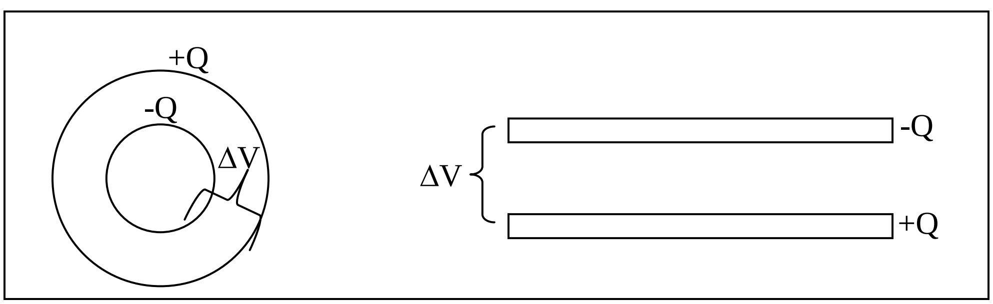
Capacitance and Resistance

Capacitance

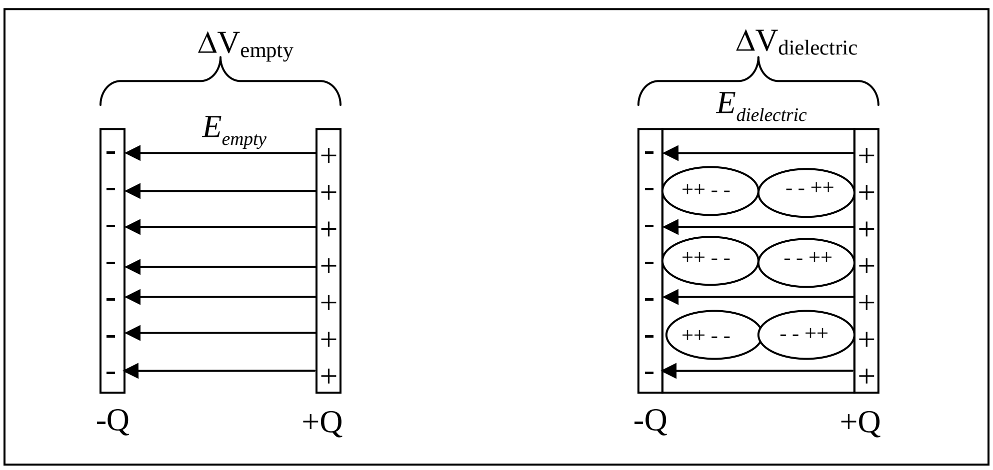
In the last chapter we studied charges and moving charges around in electrostatic fields. We also calculated work done in moving theses charges around. Now we will build a device useful for storing electric charge, called the capacitor. The capacitors only function is to store electric charge. A capacitor is simply two metallic plates separated by a gap, where the gap between the plates is usually filled with a material called a dielectric. What we will find is that the amount of charge that a capacitor can store is a geometric property of the capacitor. Each metallic plate stores charge of the same magnitude and of opposite sign. These capacitor plates are separated by a distance *d*, and because there is a separation of electric charge, a difference in electric potential exists across the metal plates. Some typical capacitor geometries are shown below. We have on the left two cylindrical plates while on the right we have two parallel plates (of arbitrary shape.)



We will begin our study of capacitance by investigating the geometric properties and quantifying our results through by assigning a capacitance (a relation between the electric potential created across the plates and the amount of charge on the plates.) We will then learn how to charge and discharge a capacitor as well as calculate the amount of energy stored in a fully charged capacitor. In order to investigate how a capacitor is charged and discharged we need to study the "movement" of the electric charges in an electric circuit. The movement of the electric charges around a closed path is called the electric current. However, before we go too far, we should start looking at capacitance and the storage of electric charge.

Suppose that we have two parallel plates (of arbitrary shape) with a cross sectional area A. Let these plates be separated by a distance d. We will assume to start, that we have a charge $+Q$ put on one plate and charge $-Q$ on the other plate. Experimentally we find that the potential difference that exists is proportional to the amount of charge that we have on each plate, meaning that if the charge on the plates, say doubles, so too does the electric potential difference across the plates. To quantify this result, we multiply by a constant so that $\Delta Q = C\Delta V$, where this constant of proportionality is called the capacitance, C, in units of Coulombs per Volt, also called a Farad. In exactly the same way that one Coulomb represents a huge amount of charge, so too a Farad represents a large capacitance, and usually we only see fractions of a Farad.

Earlier we mentioned that the plates are usually separated by a material, called a dielectric that is meant to keep the plates from touching and neutralizing. A dielectric material in between the plates of a capacitor affects the value of the capacitor. A dielectric is a material that is easily polarized due to the influence of the charges on the plates. This means that the molecules that make up the dielectric will be able to align themselves with the charges on the plates. So let's first assume that we have an empty (no dielectric material in between the capacitor plates) and that we charge the capacitor so that we have a magnitude of charge Q on each plate and then removed the charging device, thus ensuring that the magnitude of the charge placed cannot change. The number of electric field lines that pass through the area of the plates perpendicularly (by Gauss' Law) will be proportional to the magnitude of the electric field that exists. For an empty (or air-filled) capacitor we have the magnitude of the electric field given as E_{empty} as shown in the figure below. Inserting the dielectric, the molecules of the dielectric are polarized, meaning that their ends have had a charge induced on them and the molecule has an induced electric field with the same magnitude and opposite direction that cancels the electric field due to the charges on the capacitor plates. However, some places are not occupied by the dielectric molecules and here the electric field due to the charges on the plates is unchanged. We count the number of field lines as a measure of the magnitude of the electric field when the dielectric is inserted and we call this value $E_{dielectric}$.



We can immediately see that the number of field lines with the dielectric is less than those without the dielectric and we define the ratio of the E_{empty} to $E_{dielectric}$ as the dielectric constant $\kappa = \frac{E_{empty}}{E_{dielectric}}$, or the amount by which the number of electric field lines changes. Since the number of field lines in the empty (or air-filled) capacitor is greater than the number of field lines when the dielectric is inserted, dielectric constant are always greater than unity. The electric field can be related to the potential difference that exists across the capacitor plates, both of which are separated by a distance d. We have $E_{empty} = \frac{\Delta V_{empty}}{d} = \kappa E_{dielectric} = \kappa \frac{\Delta V_{dielectric}}{d}$ so that the ratio of the potential differences is $\kappa = \frac{\Delta V_{empty}}{\Delta V_{dielectric}}$. From our definition of capacitance, and knowing that the amount of charge has to remain constant since we disconnected the charging device, we have that $\kappa = \frac{\Delta V_{empty}}{\Delta V_{dielectric}} = \frac{Q/C_{empty}}{Q/C_{dielectric}} \rightarrow C_{dielectric} = \kappa C_{empty}$, or that the capacitance with the dielectric material inserted is greater than that without a dielectric material. If a charging device were attached to the capacitor with the dielectric, more charge would flow onto the plates, because we have a greater capacity for storing charge, mainly due to the reduce electric field that is established between the plates. In other words, it is easier to put the more charges on the plates and ultimately more charges would flow.

We could measure potential differences for various amounts of charge placed on each plate in order to perhaps use graphical analysis to determine the capacitance of the system. However, is there a way to evaluate the capacitance that avoids this procedure? The answer is of course, yes. To theoretically determine the capacitance of a system of two parallel plates of cross sectional area *A*, separation *d*, and with magnitude of charge on each plate *Q*, we turn to Gauss' Law. Using Gauss Law to evaluate the electric field that exists between the plates we find $E = \frac{Q}{\varepsilon_0 A}$, where *A* is the cross-sectional area of the plates. For the empty (or air-filled) capacitor, we have the magnitude of the electric field given as $E_{empty} = \frac{Q}{\varepsilon_0 A}$. Returning to the relation between the electric fields of the empty and dielectric filled capacitors we have $E_{dielectric} = \frac{E_{empty}}{\kappa} = \frac{\Delta Q}{\kappa\varepsilon_0 A} = \frac{\Delta V_{dielectric}}{d}$. Rearranging this expression we find that

$\Delta Q = \frac{\kappa\varepsilon_0 A\Delta V_{dielectric}}{d} = \kappa C_{empty}\Delta V_{dielectric} = C_{dielectric}\Delta V_{dielectric}$, where the capacitance of an empty (or air-filled) capacitor is $C_{empty} = \frac{\varepsilon_0 A}{d}$. Here we can see that the capacitance depends on the geometric properties out of which the capacitor is constructed. This result is valid for any type of capacitor, not just for parallel plates, although the analysis to get this result in other geometries is harder. Further, we can see by this result that if the plate area increases then more charge can be stored, and the capacity to hold the charge increases.

When a capacitor stores charge, it also stores energy. It takes work to put the charge on the capacitor plates and the work done manifests itself as a storage of energy in the capacitor, specifically in the electric field that exists between the capacitor plates. To calculate the amount of energy that is stored, we need to calculate the amount of work done. This is an integral process analogous to placing charges one at a time across the plates until the total amount of charge that is required on each plate is achieved. Here we will calculate the amount of work, but considering some small amount of work done each time another charge q is placed in the potential created due to all other placed charges. The total work (the product of the charge and the potential difference created) then is given by $W = \int_{q=0}^{q} V dq = \int_{q=0}^{Q} \left(\frac{q}{C}\right) dq = \frac{Q^2}{2C} = \frac{1}{2} CV^2 = \frac{1}{2} QV$, where the definition of capacitance has been used.

We have said that capacitors are used to store charge and that it takes work, from some external agent (a battery perhaps) to charge the capacitor. Now we will apply what we have learned about capacitors and tackle some problems involving capacitance and the energy stored in the electric field of a capacitor.

***Example #1* – Calculating capacitance**

Two conductors have net charges of ±*10 μC* and a potential difference of *10 V* exists across the conductors. What are the capacitance of the system the potential difference that would be measured across the conductors if the charge on the conductors is increased to ±*100 μC*?

Solution – The capacitance is given through the relation

$$C = \frac{q}{V} = \frac{10 \times 10^{-6} C}{10V} = 1 \times 10^{-6} F = 1 \mu F \ .$$

Since the capacitance is a property of the arrangement of conductors it is a constant. Thus, if the charge is increased on the conductors, the new potential difference that would be measured is

$$q_{new} = CV_{new} \rightarrow V_{new} = \frac{q_{new}}{C} = \frac{100 \times 10^{-6} C}{1 \times 10^{-6} F} = 100V$$

***Example #2* – How many electrons flow?**

How many electrons flow onto a parallel plate capacitor that has a capacitance of *9.0 μF* and is connected to a battery that is rated at *12.0V*?

Solution – The amount of charge that flows is given as

$q = CV = 9.0\times10^{-6} F \times 12.0V = 1.1\times10^{-4} C$.

Since the charge on the electron is $1.6x10^{-19}$ C, we have $6.75x10^{14}$ e^- that flow.

***Example #3* – The cell membrane as a capacitor**

A cell membrane can be modeled as a capacitor. What is the magnitude of the electric field across a cell membrane if the membrane is $1.1x10^{-8}m$ thick and a resting potential difference across the cell membrane is *-70 mV*?

Solution – We find the electric field using the relation

$$E = -\frac{\Delta V}{\Delta r} = \frac{70\times10^{-3}V}{1.1\times10^{-8}m} = 6.4\times10^{6}\,\tfrac{V}{m}\,.$$

***Example #4* – Lightning**

In lightning storms, the potential difference between the Earth and the bottom of the thunderclouds can be as high as 35,000,000 V. The bottoms of the thunderclouds are typically 1500 m above the Earth, and can have an area of $110\,\text{km}^2$. Modeling the Earth-cloud system as a huge capacitor, calculate (*a*) the capacitance of the Earth-cloud system, (*b*) the charge stored in the "capacitor," and (*c*) the energy stored in the "capacitor."

Solution -

(*a*) From the definition of the capacitance, we have

$$C = \frac{\varepsilon_0 A}{d} = \frac{\left(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)\left(110\times10^{6}\,\text{m}^2\right)}{\left(1500\,\text{m}\right)} = 6.49\times10^{-7}\,\text{F} \approx \boxed{6.5\times10^{-7}\,\text{F}}$$

(*b*) From the relation between the charge and the potential difference we have

$$Q = CV = \left(6.49\times10^{-7}\,\text{F}\right)\left(3.5\times10^{7}\,\text{V}\right) = 22.715\,\text{C} \approx \boxed{23\,\text{C}}$$

(*c*) The energy stored in a capacitor is calculated using

$$\text{PE} = \tfrac{1}{2}QV = \tfrac{1}{2}\left(22.715\,\text{C}\right)\left(3.5\times10^{7}\,\text{V}\right) = \boxed{4.0\times10^{8}\text{J}}$$

***Example #5* – More on cell membranes**

Suppose a biological membrane with a specific capacitance of 1 μF/cm^2 has a resting surface charge density of 0.1 μC/cm^2. Also suppose there are 50 sodium channels per μm^2 and that when each opens for 1 ms 1000 Na$^+$ ions flow through the channel. Find the membrane voltage 1 ms after 10% of these channels open, assuming no other changes occur during this time.

Solution - Here we need to calculate the actual charge that flows per unit area. To calculate this we need

$$\frac{Q_{total}}{1\mu m^2} = \left(\frac{50_{Na^+ channels}}{1\mu m^2}\right)\times\left(\frac{1000_{Na^+ ions}}{channel}\right)\times\left(\frac{1.6\times10^{-19}C}{Na^+ ion}\right)\times 10\% = \frac{8\times10^{-16}C}{1\mu m^2}.$$

Then, the membrane voltage is simply

$$\Delta V = \frac{Q_{total}}{Area}\times\frac{Area}{C} = \frac{8\times10^{-16}C}{\mu m^2}\times\frac{1\times10^{8}\mu m^2}{1\times10^{-6}F} = 0.08V = 80mV$$

***Example #6* – Even more on cell membranes**

In a 100 μm^2 area of a muscle membrane having a density of sodium channels of 50 per μm^2 of surface area, when the sodium channels open there is a rapid flow of 1000 ions per channel across the membrane. Assuming a 100 mV resting potential, all the channels opening at once and a membrane capacitance of 1 μF/cm^2, find the voltage change across this area of membrane due solely to the sodium ion flow.

Solution - The potential across the membrane is given as the total charge that flows divided by the capacitance of the muscle membrane. The total charge that flows is given from $Q_{total} = \left(\frac{50_{Na^+ channels}}{1\mu m^2}\times 100\mu m^2\right)\times\left(\frac{1000_{Na^+ ions}}{channel}\right)\times\left(\frac{1.6\times10^{-19}C}{Na^+ ion}\right) = 8\times10^{-13}C$. Second the capacitance is product of the specific capacitance and the muscle area. This gives the capacitance of the muscle as 1x10^{-12}F. Thus the potential difference across the membrane is $\Delta V = \frac{Q_{total}}{C} = \frac{8\times10^{-13}C}{1\times10^{-12}F} = 0.8V = 800mV$ or a change of 700mV.

Example #7 – More on Capacitance

An air-spaced parallel-plate capacitor has an initial charge of 0.05 μC after being connected to a 10 V battery.

a) What is the total energy stored between the plates of the capacitor?

b) If the battery is disconnected and the plate separation is tripled to 0.3 mm, what is the electric field before and after the plate separation change?

c) What is the final voltage across the plates and the final energy stored between the plates?

d) Calculate the work done in pulling the plates apart. Does this fully account for the energy change in part (b)?

Solution -

a. The energy is given by

$$E = \tfrac{1}{2}CV^2 = \tfrac{1}{2}\left(\frac{Q}{V}\right)V^2 = \tfrac{1}{2}QV = \tfrac{1}{2}\times 0.05\times 10^{-6}\,C\times 10V = 2.5\times 10^{-7}\,J\ .$$

b. The electric field is given through $E = \dfrac{\Delta V}{\Delta d}$. The initial electric field is thus 10V / 0.1mm = 100 kN/m and the electric field after the separation is depends on the potential across the plates. To calculate this potential we know that the charge remains fixed (the battery has been disconnected) and the capacitance decreases by a factor of 3. Thus the new potential difference is give $\Delta V_{new} = \dfrac{\Delta Q}{C_{new}} = \dfrac{0.05\times 10^{-6}C}{5\times 10^{-9}F/3} = 30V$ and the electric field is 30V / 0.3mm = 100kN/m

c. The final potential is the same as the battery or 30V and the energy stored in the capacitor is $E = \tfrac{1}{2}\dfrac{Q^2}{C_{new}} = \tfrac{1}{2}\times\dfrac{\left(0.05\times 10^{-6}C\right)^2}{5\times 10^{-9}C/3} = 7.5\times 10^{-7}\,J$.

d. The work done is the change in energy which is 5x10^{-7}J.

Electric Current

In our studies up to now, we have looked at electrostatics, or electric charges that are at rest and we have investigated the electric forces and electric fields due to collections of point charges. We have also investigated, briefly, the electric forces and fields due to planes of charge and the storage of charge when we studies capacitors. Now we would like to turn our attention to the situation where we actually let the charges start to move. The movement of an electric charge is called electric current.

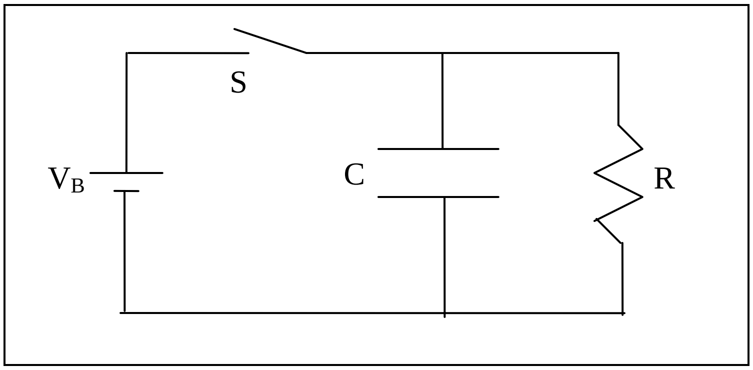
To establish a current flow in a conductor we need to establish an electric field (so that the charge carriers feel an electric force and will experience a change in motion in response to that force). Further, to establish an electric field in the conductor we will need to create a potential difference across the conductor, using perhaps a battery, a device that transforms chemical energy into electrical energy. When a continuous conducting path is created between the terminals of a battery an electric circuit is created and an electric current will flow. The amount of electric charges that pass through the wires (or conductors) cross section at any point per unit time is called the electric current. Thus we have the electric current $I = \frac{\Delta Q}{\Delta t}$ in units of the Ampere, or one Coulomb of charge that passes any point in one second of time. Hence, since the Coulomb is a large unit of charge, an Ampere is a large amount of current.

Conductors are composed of metals with many free electrons. In response to an applied electric field these electrons will move (in a direction opposite to that of the electric field that is applied.) For historical reasons, we choose not to use the electrons when developing our model of current flow in a conducting wire. Instead we adopt a conventional current of positive charge carriers, keeping in mind that the actual flow of current is due to electron flow. In addition, there are two types of current flow, in response to the nature of the applied electric field (and applied potential.) Those two currents are labeled as direct current (DC) and alternating current (AC). If the electric potential is constant in time, and hence the electric field is constant in time, the flow of conventional current will be in only one direction and will be constant in time as well. Alternately, if the electric potential oscillates in time, say in a sinusoidal fashion, the electric field will oscillate sinusoidally in time and the current will fluctuate sinusoidally with time, and this is termed alternating current.

Resistance and Electric Circuits

Charging and Discharging Capacitors

Having investigated how capacitors function, as well as how charges move in conductors, the electric current, the question that we avoided was actually how we get the charge onto the capacitor or even how we take the charge off of the capacitor. In order to charge and discharge a capacitor, we will assume a circuit that contains a battery connected in parallel (using a switch) with a resistor and a capacitor connected in series as shown below. When the switch is closed current will flow through the resistor and build up on the capacitor as a function of time. When the switch is opened current will from through the resistor and will eventually neutralize the capacitor.



Charging a Capacitor

To study the charging of the capacitor we apply Kirchhoff's rule for potentials to the circuit containing the battery, capacitor and resistor. Traversing the closed loop one time we have $V_B - V_R - V_C = V_B - IR - \frac{q}{C} = V_B - \frac{dq}{dt}R - \frac{q}{C} = 0$. This is a differential equation that when solved will show how the charge builds up on the capacitor as a function of time. To solve this equation, we "separate and integrate" meaning that we will place all of the terms involving the charge (one of the variables we want to solve for) on one side of the equals sign and the time dependant terms (the other variable of interest) one the other side. Rewriting our equation for the charging of a capacitor we have $\frac{dq}{(V_B C - q)} = \frac{dt}{RC}$. This expression will be integrated from time $t_i = 0$ (when $q_i = 0$) to a later time t_f (when charge q_f has been placed on the capacitor.) Thus we have $\int_{q_i=0}^{q_f} \frac{dq}{(V_B C - q)} = \int_{t_i=0}^{t_f} \frac{dt}{RC}$ which evaluates to $q_f(t) = q_{\max}\left(1 - e^{-\frac{t}{RC}}\right)$, where the maximum charge on the capacitor is a product of the potential difference of the battery and the capacitance of the capacitor, $V_B C$.

Here we encounter a term that we will use quite extensively to characterize resistor-capacitor circuits. The product of the resistance and the capacitance is called the time constant, $\tau = RC$, of the circuit, in units of seconds. To see physically what the time constant means, let's insert it into the equation for the charge accumulation as a function of time. Thus we have $q_f(RC) = q_{\max}\left(1 - e^{-\frac{RC}{RC}}\right) = 0.632 q_{\max}$, or 63.2% of the maximum amount of charge has been put on the capacitor. Therefore we conclude that in one time constant, we accumulate 63.2% of the maximum amount of charge.

However, one might ask how do you actually count or "see" the charges as they get put on the capacitor? Of course we cannot, but what we can "see" is how the potential difference across the capacitor changes with time using a voltmeter or a computer sensor designed to measure voltages. Since we have a relation between the potential difference across a capacitor and the charge on the capacitor $q = CV$, we can rewrite the equation for the accumulation of charge on the capacitor in terms of the potential difference, and we have $V_f(t) = V_{\max}\left(1 - e^{-\frac{t}{RC}}\right)$.

Discharging a Capacitor

If we disconnect the battery from the circuit by opening the switch, the capacitor will discharge through the resistor. To describe how the charge leaves the capacitor as a function of time we return to our original equation as derived by using Kirchhoff's equation or, $V_B - V_R - V_C = V_B - IR - \frac{q}{C} = V_B - \frac{dq}{dt}R - \frac{q}{C} = 0$ and set the potential difference due to the battery, V_B, equal to zero as we have removed it from the circuit. This gives $-IR-\frac{q}{C}=-\frac{dq}{dt}R-\frac{q}{C}=0$. We again have a differential equation involving the charge q and time. We again "separate and integrate" this equation from time $t_i = 0$ (when $q_i = q_{max}$) to a later time t_f (when charge q_f remains on the capacitor.) Thus we have $\int_{q_i=q_{\max}}^{q_f}\frac{dq}{q}=\int_{t_i=0}^{t_f}-\frac{dt}{RC}$ which evaluates to $q_f(t)=q_{\max}\left(e^{-\frac{t}{RC}}\right)$.

Again we have the product of the time constant in the denominator of the exponential function. Evaluating at this time we find $q_f(RC)=q_{\max}\left(e^{-\frac{RC}{RC}}\right)=0.368q_{\max}$, or 36.8% of q_{max} remains on the capacitor. This of course means that 63.2% of the maximum charge has been removed from the capacitor. Therefore we can conclude that the *RC*-time constant is the time required to charge the capacitor to 63.2% of its maximum charge *OR* the time required to remove 63.2% of the maximum accumulated charge from the capacitor.

However, we could ask how do you actually count or "see" the charges as they leave the capacitor? We express the answer to this in terms of the potential difference across the capacitor as it changes with time using a voltmeter or a computer sensor designed to measure voltages. Again, using the relation between the potential difference across a capacitor plates and the charge on the capacitor $q = CV$, we write for a discharging capacitor $V_f(t)=V_{\max}\left(e^{-\frac{t}{RC}}\right)$.

***Example #8* – A Simple RC Circuit**

A 100 μF capacitor wired in a simple series *RC* circuit is initially charged to 10 μC and then discharged through a 10 kΩ resistor.

a. What is the time constant of the circuit?

b. What is the initial current that flows?

c. How much charge is left on the capacitor after 1 time constant?

d. What is the current after 1 time constant?

e. How much charge is left on the capacitor after 3 time constants have elapsed and what current is flowing then?

Solution -

a. The time constant is the product of the resistance and the capacitance, or 1.0s.

b. The initial current that flows is found from Ohm's Law

$$V_C = \frac{Q}{C} = IR \to I = \frac{Q}{RC} = \frac{10\times10^{-6}C}{1s} = 10\times10^{-6}A = 10\mu A .$$

c. After 1 time constant we have $Q(1s) = Q_o e^{-\frac{1s}{1s}} = \frac{10\mu C}{e} = 0.368\times10\mu C = 3.68\mu C$.

d. The current that flows after one time constant is

$$I(1s) = I_o e^{-\frac{1s}{1s}} = \frac{10\mu A}{e} = 0.368\times10\mu A = 3.68\mu A$$

e. After 3 time constants we have

$Q(3s) = Q_o e^{-\frac{3s}{1s}} = \frac{10\mu C}{e^3} = 0.050\times10\mu C = 0.50\mu C$ and the current that is flowing is

$$I(3s) = I_o e^{-\frac{3s}{1s}} = \frac{10\mu A}{e^3} = 0.050\times10\mu A = 0.50\mu A .$$

***Example #9* – The Defibrillator**

The immediate cause of many deaths is ventricular fibrillation, an uncoordinated quivering of the heart as opposed to proper beating. An electric shock to the chest can cause momentary paralysis of the heart muscle, after which the heart will sometimes start organized beating again. A *defibrillator* is a device that applies a strong electric shock to the chest over a time interval of a few milliseconds. The device contains a capacitor of several microfarads, charged to several thousand volts. Electrodes called paddles, about 8 cm across and coated with conducting paste, are held against the chest on both sides of the heart. Their handles are insulated to prevent injury to the operator,

who calls "Clear!" and pushes a button on one paddle to discharge the capacitor through the patient's chest. Assume that an energy of 300 J is to be delivered from a 30.0-μF capacitor.

a. To what potential difference must it be charged?

Consider the following, a *defibrillator* connected to a 32 μF capacitor and a 47 kΩ resistor in a RC circuit. The circuitry in this system applies 5000 V to the RC circuit to charge it.

b. What is the time constant of this circuit?

c. What is the maximum charge on the capacitor?

d. What is the maximum current in the circuit during the charging process?

e. What are the charge and current as functions of time?

f. How much energy is stored in the capacitor when it is fully charged?

Solution –

a. The energy stored in this system is given from $U = \frac{1}{2} C \Delta V^2$ so that the potential difference is $\Delta V = \sqrt{\frac{2U}{C}} = \sqrt{\frac{2(300\text{ J})}{30 \times 10^{-6}\text{ C/V}}} = \boxed{4.47 \times 10^3\text{ V}}$

b. The time constant is given as $\tau = RC = 47 \times 10^3 \Omega \times 32 \times 10^{-6} F = 1.5s$.

c. The maximum charge on the capacitor is the product of the capacitance and the maximum voltage, which is

$$Q_{\max} = CV_{\max} = 32 \times 10^{-6} F \times 5000V = 0.160C$$

d. The maximum current is given from Ohm's Law

$$I_{\max} = \frac{V_{\max}}{R} = \frac{5000V}{47 \times 10^3 \Omega} = 0.106A.$$

e. The charge and currents are thus given as $Q(t) = 0.160C\left(1 - e^{-\frac{t}{1.5s}}\right)$ and $I(t) = 0.106A\left(1 - e^{-\frac{t}{1.5s}}\right)$.

f. The maximum amount of energy stored when the capacitor is fully charged is $E_{\max} = \frac{1}{2} CV_{\max}^2 = \frac{Q_{\max}^2}{2C} = \frac{1}{2} Q_{\max} V_{\max} = 400J$

***Example #10* – How Many Lightning Strikes in a typical day**

The Earth's atmosphere is able to act as a capacitor, with one plate the ground and the other the clouds and in between the plates an air gap. Air, however, is not a perfect insulator and can be made to conduct, so that the separation of charges from the cloud to ground can be bridged. Such an event is called a lightning strike. For this question, we'll model the atmosphere as a spherical capacitor and try to calculate the number of lightning strikes that happen every day. First, let's assume that the clouds are distributed around the entire Earth at a distance of 5000 m above the ground of area $4pR^2_{Earth}$, where R_{Earth} = 6400km.

a. What is the resistance of the air gap?

Next we need to calculate the capacitance of the Earth-cloud capacitor. Here we will use the fact that the capacitance $C = \frac{Q}{\Delta V}$ and we'll calculate ΔV. Assuming that we have a spherical charge distribution $V = k\frac{Q}{r}$, so that ΔV is the difference in potential between the lower plate (the Earth's surface) and the upper plate (the clouds) and that in a typical day, $5x10^5$ C of charge is spread over the surface of the Earth.

b. What is ΔV?

c. What is the capacitance of the Earth-cloud capacitor?

Since the accumulated charge will dissipate through the air, we have a simple RC circuit.

d. What is the time constant for this discharge that is spread over the whole surface of the earth?

Experimentally it is found that for each lightning strike about 25 C of negative charge is delivered to the ground.

e. What is the number of lightning for this amount of charge?

f. Approximately how long would it take the Earth-cloud capacitor to discharge to 0.3% of its initial amount?

So to answer the question of how many lightning strikes per day, we know that we get part e number of strokes in part f amount of time.

g. How many lightning strokes per day?

Solution -

a. The resistance of the air-gap is given

as $R = \rho \frac{L}{A} = \left(3\times10^{13}\Omega m\right)\frac{5\times10^{3}m}{4\pi\left(6.4\times10^{6}m\right)^{2}} = 291\Omega$.

b. Assuming that the Earth-cloud system can be modeled as two spherical charge distributions separated by a fixed distance, we find the difference in potential to be

$$\Delta V = V_{lower} - V_{upper} = k\left(\frac{Q}{r_{lower}} - \frac{Q}{r_{upper}}\right) = 9\times10^{9}\,\tfrac{Nm^2}{C^2}\times5\times10^{5}C\times\left(\frac{1}{r_{earth}} - \frac{1}{r_{earth}+5000m}\right) = 4.88\times10^{5}V.$$

c. The capacitance is therefore $C = \frac{Q}{\Delta V} = \frac{5\times10^{5}C}{4.88\times10^{5}V} = 1.03F$.

d. The time constant is the product of the resistance and capacitance of the circuit. Thus the time constant is $\tau = RC = 291\Omega\times1.03F = 298.2s$.

e. For each lightning strike, 25C of charge is delivered. Thus the number of lightning strikes is the total charge divided by 25 C per strike, or 20,000 strikes.

f. To calculate the time, we use the equation for a discharging capacitor.

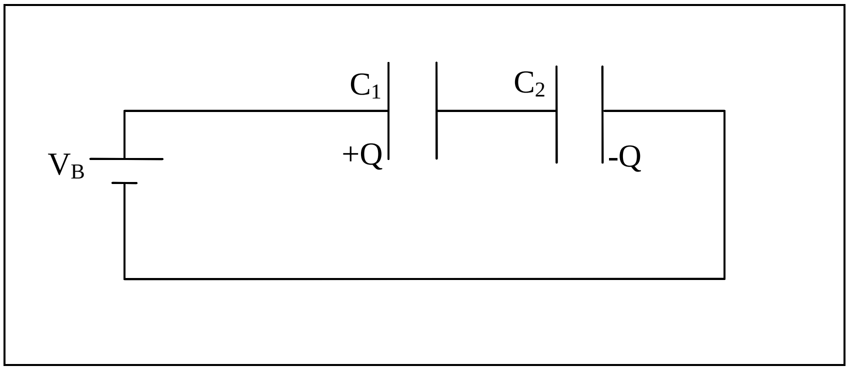
$$Q(t) = Q_{\max}e^{-\frac{t}{RC}} \rightarrow 0.003 = e^{-\frac{t}{298.2s}} \rightarrow t = 1732.3s = 0.48hr.$$

g. So we get 20,000 strikes every 0.48hrs, or 41,667 strikes per hour. Thus in 24 hours we get about 1,000,000 strikes. This value, even if only approximate, is not far from the observed number of about 1 million per day!!

Capacitors in Series and Parallel

Capacitors in Series

In an analogy with resistor circuits, we could place capacitors in combination and calculate the equivalent capacitance of the circuit. As or first problem, we will place two capacitors, with capacitances C_1 and C_2 in series with each other and this combination in series with a single battery as shown in the diagram below.

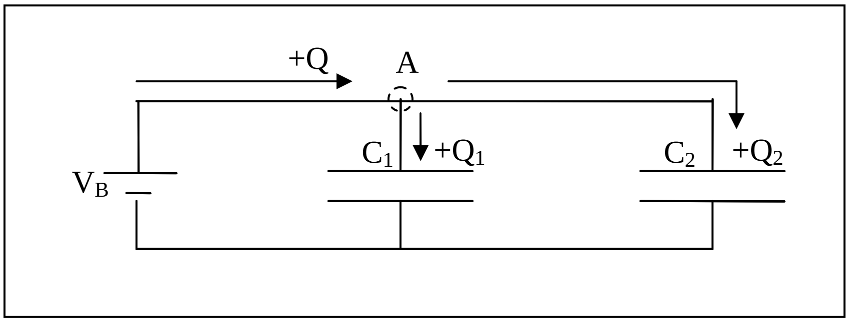


To calculate the equivalent capacitance of the capacitors in series we recognize that a time dependant current will flow and a charge of magnitude +*Q* will accumulate on the leftmost capacitor plate of C_1 which is connected to the positive terminal of the battery. An equal magnitude of negative charge will then appear on the rightmost capacitor plate of C_2 because it is connected to the negative terminal of the battery. In between these two extremities, the circuit has to remain uncharged since the inner two plates are connected by a wire, and any charge on those plates would immediately neutralize each other. Thus we have in essence one large capacitor formed by the outer most plates. The potential difference across the combination, by Kirchhoff's Rule, must sum to the potential difference of the battery. Thus we have $V_B = V_{C_1} + V_{C_2} = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$ for the circuit. Defining the inverse of the equivalent capacitance to be $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$ for two capacitors wired in series. If there were more than two capacitors in series we could extend the analysis and we would find that for *N* capacitors wired in series the equivalent capacitance is simply $\frac{1}{C_{eq}} = \sum_{i=1}^{N} \frac{1}{C_i}$.

Thus we see, in analogy with resistors, that capacitors in series add like resistors in parallel.

Capacitors in Parallel

Of course we could have wired these two capacitors in parallel with each other. Suppose, now that we place the two capacitors, with capacitances C_1 and C_2 in parallels with each other and this combination in parallel with a single battery as shown in the diagram below.



To calculate the equivalent capacitance of the capacitors in series we recognize that a time dependant current will flow, but at any interval of time, a total charge $+Q$ will arrive at the junction labeled A, and a charge Q_1 will be deposited on capacitor C_1 while charge Q_2 will be placed on capacitor C_2, such that $Q = Q_1 + Q_2$. However we know that elements in parallel have the same potential drops across them, so the potential drop across capacitor C_1 is simply the potential difference of the battery, as is the case for the potential drop across capacitor C_2. Using the fact that the potential drops across the capacitors is the same and that the charge entering the junction A must equal the sum of charge leaving the junction, we have $Q = Q_1 + Q_2 = C_1V_1 + C_2V_2 = (C_1 + C_2)V$. Again we see that if two capacitors are in parallel we define the equivalent capacitance to be the sum of the individual capacitances, $C_{eq} = C_1 + C_2$. If there were more than two capacitors in parallel we could redo the analysis and we would find that for N capacitors wired in parallel the equivalent capacitance is simply $C_{eq} = \sum_{i=1}^{N} C_i$. Thus we see, in analogy with resistors, that capacitors in parallel add like resistors in series.

Example #11 **- An Example of Capacitors in Series and Parallel**

A simple *RC* series circuit has a 100 μF capacitor.

a. If the time constant is 50 s, what is the value of the resistor?
b. Suppose that a second identical resistor is inserted in series with the first. What is the new time constant of the circuit?
c. Suppose the second identical resistor is placed in parallel with the first resistor, still connected to the capacitor. What is the new time constant in this case?
d. Suppose that we use the single resistor from part a, but now a second identical capacitor is connected in series with the first. What is the new time constant of the circuit?
e. Suppose that we use the single resistor from part a, but now a second identical capacitor is connected in parallel with the first. What is the new time constant of the circuit?

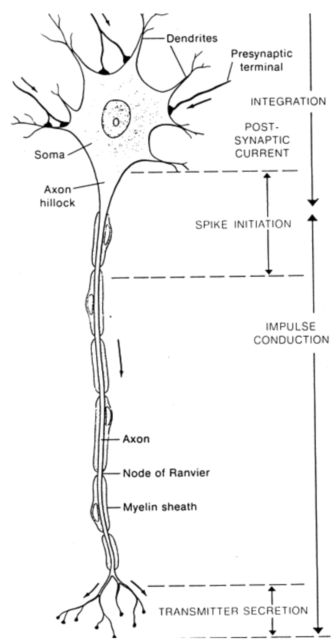
Solution –

a. The resistance is given from $\tau = RC \rightarrow R = \frac{\tau}{C} = \frac{50s}{100\times10^{-6}F} = 5\times10^{5}\Omega$.
b. The equivalent resistance for these two resistors in series is the sum of the individual resistances, or $1\text{x}10^6$ Ω and the new time constant is $\tau_{series} = R_{series}C = 1\times10^{6}\Omega\times100\times10^{-6}F = 100s$.
c. The equivalent resistance for these two resistors in parallel is the reciprocal of the sum of the reciprocals of the individual resistances, or $2.5\text{x}10^5$ Ω and the new time constant is $\tau_{parallel} = R_{parallel}C = 2.5\times10^{5}\Omega\times100\times10^{-6}F = 25s$.
d. The equivalent capacitance for these two capacitors in series is the reciprocal of the sum of the reciprocals of the individual capacitances, or $5.0\text{x}10^{-5}$ F and the new time constant is $\tau_{series} = RC_{series} = 5.0\times10^{5}\Omega\times5.0\times10^{-5}F = 25s$.

e. The equivalent resistance for these two capacitors in parallel is the sum of the individual capacitances, or 200x10^{-6} Ω and the new time constant is $\tau_{parallel} = RC_{parallel} = 5.0\times10^{5}\Omega\times200\times10^{-6}F = 100s$.

The Cell

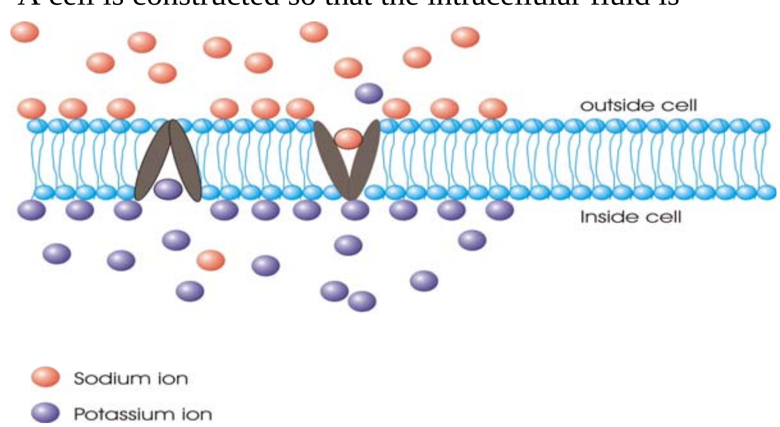
As one final example, we will investigate the cell as a model of electrical conduction. Flow of charge in the human nervous system gives us a means of investigating the world around us, communication in the body, as well as control of movement of the body's systems. In the treatment that follows we will make some simplifying assumptions so that we can apply our physical principles to understand conduction within the nervous system, knowing that the complete description is very complex and beyond the scope of this book. To start discussing electrical conduction, we need to have a structure that will propagate the electric signal and this structure is the neuron. A simplified sketch of a neuron is shown on the right.



Neurons are living cells of unusual shape. There are three types of neurons. Sensory neurons bring electric signals from or to the extremities (hands, eyes, skin, etc.) and to the central nervous system. Motor neurons bring signals from the central nervous system to muscles and cause the muscle to contract, while the interneuron transmits signals from neuron to neuron. Information comes in to the neuron via synaptic connections at the dendrites (or by some electrical, chemical, or physical stimulus.) The cell decides whether or not to send an action potential down the length of the neuron, along the axon. The axon of neurons may be myelinated or unmyelinated and this affects how the electrical signal propagates down the axon, or the body of the neuron. The neuron shown above is myelinated, or the axon is covered with a myelin sheath, except at special points called the Nodes of Ranvier. The signal propagates from the dendrites down the axon and the signal is transmitted across a synapse with the aid of chemicals called neurotransmitters, which will not be discussed.

The neuron is made up of cells and before any signal propagation is initiated, the cell and the neuron are in a resting state. A stimulus is needed to start the signal propagation from cell to cell down the axon. Let's take a look at a typical cell and see how information is processed and how this information can be transmitted from cell to cell. This process is called the firing of an action potential, and will eventually be propagated along the axon of the neuron.

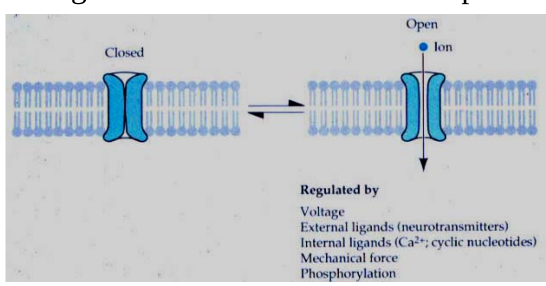
In order for an action potential to be initiated, the stimulus has to be above a certain intensity or threshold. A cell is constructed so that the intracellular fluid is separated from the extracellular fluid by a membrane, usually formed of proteins. This cell membrane is semi-permeable to different sizes



of ions and the charges are exchanged across the cell wall by a mechanism called the sodium-potassium pump, which is a protein that spans the membrane. This mechanism is complicated, but the sodium-potassium pump uses energy in the form of ATP (adenosine tri-phosphate) to move ions across the membrane against their concentration gradient. A concentration gradient exists due to concentration differences that exist across the cell wall.   This particular pump, the Sodium-Potassium ATPase, pumps two potassium ions into the cell for every three sodium ions it pumps out. The actual mechanism will not be investigated at any deeper level than what has just been presented. We will make references to the sodium-potassium pump when we talk about signal propagation down the axon, from cell to cell.

The cells are surrounded by many ions that are in an electrically neutral fluid and these ions are able to diffuse into and out of the cell across the cell membrane.   The main ions that are found in the cellular fluid are usually sodium (Na^+), potassium (K^+) and chlorine (Cl^-,) and is shown below. There are large differences in the concentrations of each ion and these concentration gradients produce a potential difference across the cell membrane and these potentials, called resting potentials $\Delta V_{resting} = V_{inside} - V_{outside}$, are usually in the range of -60 mV to -90 mV. Typical intra- and extracellular concentrations are given in the table below. The cell membrane is selectively permeable with chlorine ions and to a lesser degree potassium ions able are able to diffuse across the membrane, while sodium ions cannot flow.   From the table of ion concentrations, we see that on average, the potassium ions will tend to diffuse out of the cells in the direction of the concentration gradient, while the chlorine ions will tend to diffuse into the cell. The cellular fluid is electrically neutral and the charges tend to attach themselves to the cell wall due to their mutual electrostatic attraction across the membrane, which is on the order of 10^{-8}m in thickness. This produces a dipole layer across the cell membrane and an associated electric field on the order of $E = -\frac{\Delta V}{\Delta r} = \frac{70\times 10^{-3} V}{1\times 10^{-8} m} = 70\times 10^{5}\ \frac{V}{m}$ pointing into the cell. As charge accumulates on the inner and outer layers of the cell membrane, it becomes harder for charge to flow across the membrane and equilibrium is reached when the tendency to diffuse due to the concentration gradient is balance by the potential difference across the cell. In cells we see that the higher the concentration gradient that exists across the cell membrane, the higher the resting potential difference that will be measured across the cell.

In response to a stimulus, the cell can conduct ions across the cell membrane and this signal can be propagated from cell to cell along the axon, and the signal that propagates is called an action potential, and is a rapidly changing potential difference across the cell membrane, shown below. A stimulus will cause the sodium-potassium pump to operate and sodium channels will open in the cell membrane in response to the stimulus. These channels allow ions to pass through the membrane and can be opened and closed (called gating) through one of several different mechanisms as shown on the right. Since there is a higher concentration of sodium ion outside of the cell wall, sodium will tend to diffuse into the cell. The electrical potential measured across the cell wall changes to approximately +40 mV, and this depolarizes a local region across the cell wall. Depolarization causes potassium channels to immediately open and potassium ion diffuse out of the cell. This reestablishes the initial ion concentration and electrical potential of approximately -60 mV and repolarizes the cell, and during this time, about one millisecond, the sodium and potassium channels cannot be opened by a



stimulus. This single action potential acts as a stimulus to neighboring proteins (and eventually the neighboring cells) and initiates an action potential in another part of the neuron, due to the changing potential in the neighboring portions of the cell.   Thus we can also define one action potential as the change in potential difference across the cell membrane from its resting state to the cell firing (rapidly depolarizing and repolarizing) and returning to its resting state.

This will ultimately produce a wave of action potentials that travel from cell to cell (depolarizing and repolarizing the cell) from the dendrites all the way to the axon terminals, where at the axon terminal the electrical impulse is converted to a chemical signal.   This chemical or neurotransmitter crosses the synapse between adjacent neurons and initiates an action potential on another neuron. The action potential activates a calcium channel and doubly ionized calcium (Ca++) diffuses into the neuron. The presence of the calcium ion causes vesicles to fuse with the cell membrane and through a process called exocytosis, neurotransmitters (the chemicals) are released into the synapse. The neurotransmitters will diffuse across the synapse and bind to receptors cells on another neuron and this will in turn causes special sodium channels to open and an action potential is initiated in the next neuron, and we conduct an electrical signal from neuron to neuron.

Now, we have seen how the cells and neurons can conduct an electrical signa. Next we would like to calculate the capacitance of a typical neuron (since we have a separation of charge across cells of the neuron) and try to estimate how fast a signal can propagate along an axon.

Example #12 – The capacitance of an axon

Suppose that we have an axon that is *10 cm* long and has a radius of $10\ \mu m$. Given a typical membrane thickness of about $10^{-8}\ m$ and a dielectric constant for cellular fluid of *3*, what is the capacitance of the axon?

Solution – Assuming that the axon can be modeled as a cylindrically shaped capacitor with equal and oppose charges on each side, we have the plate thickness of $d = 1x10^{-8}\ m$. The cross sectional area of the plates is given as the product of the circumference of the axon and the length of the axon $A = 2\pi rl = 2\pi \times 10 \times 10^{-6}\,m \times 0.1m = 6.28 \times 10^{-6}\,m^2$. We calculate the capacitance from

$$C = \frac{\kappa\varepsilon_0 A}{d} = \frac{3 \times 8.85 \times 10^{-12}\,\frac{C^2}{Nm^2} \times 6.28 \times 10^{-6}\,m^2}{1 \times 10^{-8}\,m} = 1.7 \times 10^{-8}\,F\ .$$

Example #13 – Concentration changes due to an action potential

Suppose that we use the axon in Example #12, by what factor does the concentration of sodium ions in the cell change due to the action potential firing?

Solution – We assume from the reading that the potential change from the resting state (-*70 mV*) to the active state when the action potential fires (*+40 mV*) to calculate the amount of charge that is moved across the cell membrane. Thus we have a difference in potential of *110 mV* and using the capacitance calculated in Example #7, we have for the amount of charge $Q = CV = 1.7 \times 10^{-8}\,F \times 110 \times 10^{-3}\,V = 1.87 \times 10^{-11}\,C$. The charge on a singly ionized sodium atom is $1e^-$ and dividing the amount of charge that flows by one elementary charge we have the number of sodium ions that flows as $1.87 \times 10^{-11}\,C \times \frac{1e^-}{1.6 \times 10^{-19}\,C} = 1.2 \times 10^8\,Na^+$. Next we need to know the initial concentration of sodium ions in the cell. Looking at table above, we see *15 mol/m*^{3} for the concentration of sodium ions inside of the cell. Thus we have

$$15\,\tfrac{\text{mol}}{m^3}\times\frac{6.02\times10^{23}\,Na^+}{1\text{mol}}9.03\times10^{24}\,\tfrac{Na^+}{m^3}\ \text{ in } \pi r^2 l=\pi\times\left(10\times10^{-6}\,m\right)^2\times0.1m=3.14\times10^{-11}\,m^3$$

(the volume of the neuron), or a concentration of

$3.14\times10^{-11}\,m^3\times9.03\times10^{24}\,\tfrac{Na^+}{m^3}=2.84\times10^{14}\,Na^+$. The change in the concentration of the sodium ions is $\dfrac{1.2\times10^{8}\,Na^+}{2.83\times10^{14}\,Na^+}\approx4\times10^{-7}$, or a change of *4 parts in 10 million*, probably a concentration change that would not be measurable.

Example #14 **– The current flow across a Cell Membrane due to Na+ Flow**

How much current flows during one action potential for the concentration calculated in example #13 if the action potential lasts for 1 millisecond?

Solution – The current is the charge that flows divided by the amount of time it takes the charge to flow across the cell wall. Thus we have

$$I=\frac{\Delta Q}{\Delta t}=\frac{1.87\times10^{-11}C}{1\times10^{-3}s}=19\times10^{-9}A=19nA$$

Example #15 **– Propagation Speed along an Axon**

A neuron is stimulated with an electric pulse. The action potential is detected at a point 3.40 cm down the axon 0.0052 s later. When the action potential is detected 7.20 cm from the point of stimulation, the time required is 0.0063 s. What is the speed of the electric pulse along the axon and why are two measurements needed instead of only one?

Solution - The speed is the change in position per unit time.

$$v = \frac{\Delta x}{\Delta t} = \frac{7.20 \times 10^{-2}\,\text{m} - 3.40 \times 10^{-2}\,\text{m}}{0.0063\,\text{s} - 0.0052\,\text{s}} = \boxed{35\,\text{m/s}}$$

We need two measurements because there may be a time delay from the stimulation of the nerve to the generation of the action potential.

***Example #16* – Energy Transmission in an Action Potential**

How much energy is required to transmit one action potential along the axon of Example #12, where the energy to transmit one pulse is equivalent to the energy stored by charging the axon capacitance? What minimum average power is required for 10^4 neurons each transmitting 100 pulses per second?

Solution - The energy required to transmit one pulse is equivalent to the energy stored by charging the axon capacitance to full voltage. In Example #12, the capacitance is approximately at $1.0\times10^{-8}\,\text{F}$, and the potential difference is about 0.1 V.

$$E = \tfrac{1}{2}CV^2 = 0.5\left(10^{-8}\,\text{F}\right)\left(0.1\,\text{V}\right)^2 = \boxed{5 \times 10^{-11}\,\text{J}}$$

The power is the energy per unit time, for 10,000 neurons transmitting 100 pulses each per second. $P = \dfrac{E}{t} = \dfrac{\left(5 \times 10^{-11}\,\text{J/neuron}\right)\left(1.0 \times 10^{4}\,\text{neurons}\right)}{0.01\,\text{s}} = \boxed{5 \times 10^{-5}\,\text{W}}$

***Example #17* – The Power in the Sodium Pump**

During an action potential, Na^+ ions move into the cell at a rate of about $3\times10^{-7}\,\text{mol/m}^2\cdot\text{s}$. How much power must be produced by the “active Na^+ pumping” system to produce this flow against a $+30\text{-mV}$ potential difference? Assume that the axon is 10 cm long and $20\,\mu\text{m}$ in diameter.

Solution - The power is the work done per unit time. The work done to move a charge through a potential difference is the charge times the potential difference. The charge density must be multiplied by the surface area of the cell (the surface area of an open tube, length times circumference) to find the actual charge moved.

$$P = \frac{W}{t} = \frac{QV}{t} = \frac{Q}{t}V$$

$$= \left(3 \times 10^{-7} \frac{\text{mol}}{\text{m}^2 \cdot \text{s}}\right)\left(6.02 \times 10^{23} \frac{\text{ions}}{\text{mol}}\right)\left(1.6 \times 10^{-19} \frac{\text{C}}{\text{ion}}\right)(0.10\,\text{m})\pi\left(20 \times 10^{-6}\,\text{m}\right)(0.030\,\text{V})$$

$$= \boxed{5.4 \times 10^{-9}\,\text{W}}$$