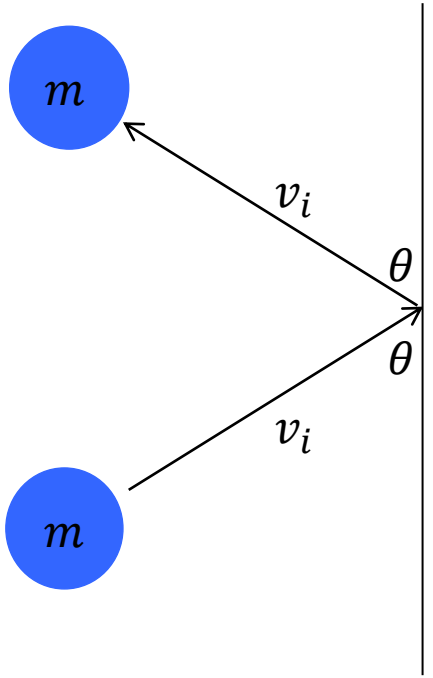
Momentum and Collisions

Example 1:

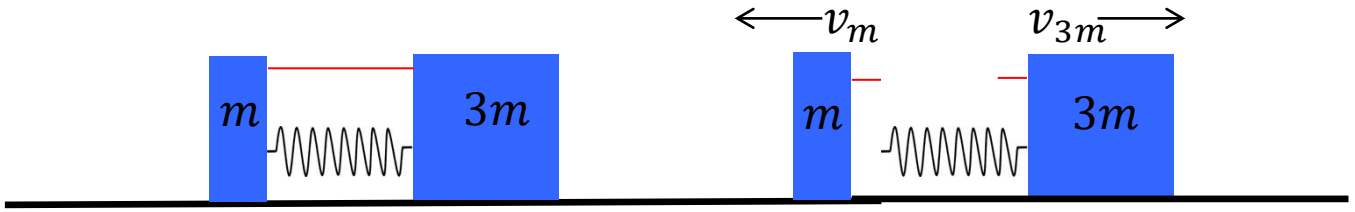
A ball of mass $m = 200g$ is thrown at a wall at an angle of $\theta = 60^0$, measured with respect to the vertical with a speed of $5\frac{m}{s}$. The ball bounces off the wall with approximately the same speed as it struck the wall.



- What are the components of the change in momentum of the ball's momentum?
- What is the change in momentum of the ball?
- If the ball is in contact with the wall for a time $\Delta t = 0.2s$, what force does the wall exert on the ball?

Example 2:

Consider the following system that has two blocks of masses m and $3m$ connected by a very light string. The masses m and $3m$ are sitting on a horizontal frictionless surface and a light (i.e., massless) spring is attached to one of the blocks and the blocks are squeezed together and tied by a light cord. If the cord is cut and the block of mass $3m$ moves to the right at a speed of $v_{3m} = 2\frac{m}{s}$, what is the speed of the block of mass m?

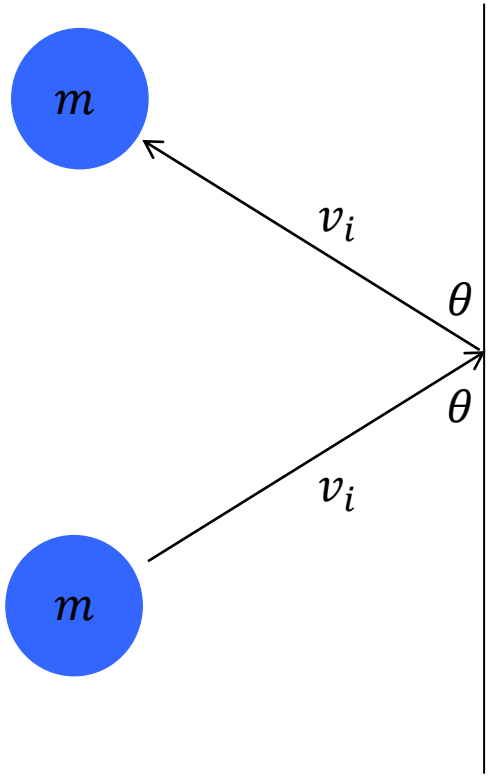


Momentum and Collisions

- Impulse-Momentum

Example 1 Solution:

- To assign directions to our vectors, let's choose a standard cartesian coordinate system.



- The change in the components of the ball's momentum:

$$\Delta p_x = mv_{fx} - mv_{ix} = -mv_i \sin\theta - (mv_i \sin\theta) = -2mv_i \sin\theta$$

$$\Delta p_x = -2mv_i \sin\theta = -2\times 0.2kg\times 5\frac{m}{2}\sin 60 = -1.73\frac{kgm}{s}$$

$$\Delta p_y = mv_{fy} - mv_{iy} = mv_i \cos\theta - (mv_i \cos\theta) = 0\frac{kgm}{s}$$

- The change in momentum of the ball:

Magnitude of the change in momentum:

$$\Delta p = \sqrt{\Delta p_x^2 + \Delta p_y^2} = \sqrt{\left(-1.73\frac{kgm}{s}\right)^2 + \left(0\frac{kgm}{s}\right)^2} = 1.73\frac{kgm}{s}$$

Direction of the change in momentum:

$$\phi = \tan^{-1}\left(\frac{\Delta p_y}{\Delta p_x}\right) = \tan^{-1}\left(\frac{0\frac{kgm}{s}}{-1.73\frac{kgm}{s}}\right) = 0$$

The change in momentum is in the negative x-direction and the angle is measured with respect to the negative x-axis. With respect to the positive x-axis $\phi = 180^0$.

Momentum and Collisions

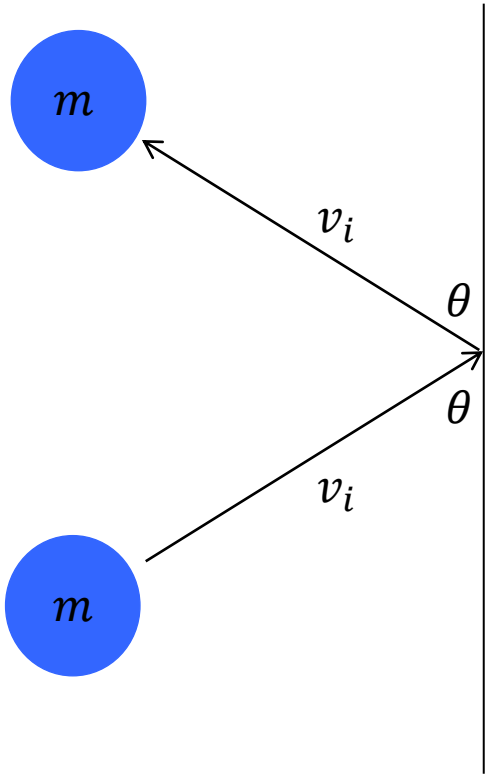
- Impulse-Momentum

Example 1 Solution continued:

- The component of the force on the ball from the wall are:

$$F_{x,bw} = \frac{\Delta p_x}{\Delta t} = \frac{-1.73\frac{kgm}{s}}{0.2s} = -8.65N$$

$$F_{y,bw} = \frac{\Delta p_y}{\Delta t} = \frac{0\frac{kgm}{s}}{0.2s} = 0N$$



- The magnitude of the force exerted on the ball by the wall is:

$$F_{bw} = \sqrt{F_{x,bw}^2 + F_{y,bw}^2} = \sqrt{\left(-8.65\frac{kgm}{s}\right)^2 + \left(0\frac{kgm}{s}\right)^2} = 8.65\frac{kgm}{s}$$

- The direction of the force exerted on the ball by the wall is:

$$\phi = \tan^{-1}\left(\frac{F_{y,bw}}{F_{x,bw}}\right) = \tan^{-1}\left(\frac{0\frac{kgm}{s}}{-8.65\frac{kgm}{s}}\right) = 0^0$$

The net force from the wall on the ball is in the negative x-direction and the angle is measured with respect to the negative x-axis. With respect to the positive x-axis $\phi = 180^0$.

Momentum and Collisions

- Impulse-Momentum

Example 1 Solution continued:

- One last thing. The ball exerts a force on the wall of equal magnitude but in the opposite direction. Thus, the wall experiences a change in momentum too.

$$F_{x,wb} = 8.65N$$

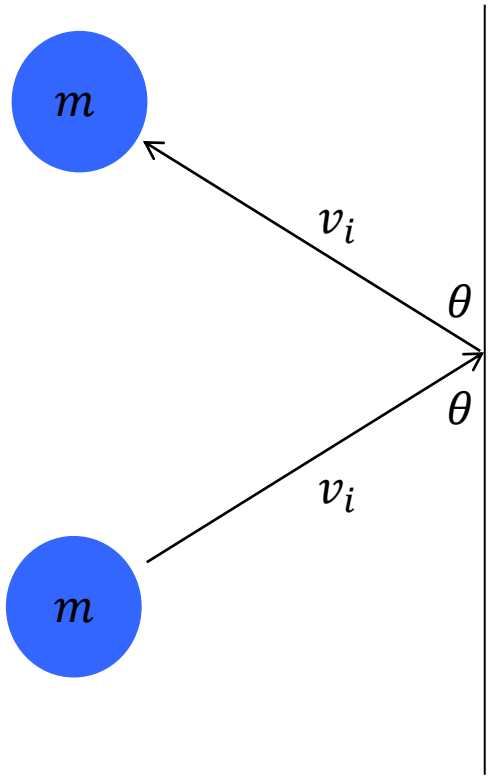
$$F_{y,wb} = 0N$$

- The magnitude of the force exerted on the ball by the wall is:

$$F = \sqrt{F_{x,wb}^2 + F_{y,wb}^2} = \sqrt{(8.65N)^2 + (0N)^2} = 8.65N$$

- The direction of the force exerted on the ball by the wall is:

$$\phi = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{0\frac{kgm}{s}}{8.65\frac{kgm}{s}}\right) = 0^0$$



- The change in momentum of the wall is in the horizontal direction (by the impulse momentum theorem).

$$\Delta p_{wb} = F_{wb}\Delta t = 8.65N \times 0.2s = 1.73\frac{kgm}{s} = m_w \Delta v_w \rightarrow \Delta v_w = \frac{1.73\frac{kgm}{s}}{m_w} \sim 0\frac{m}{s}$$

The change in speed of the wall is approximately zero. The wall is so massive compared to the ball that the wall does not move.

Momentum and Collisions

Example 2 Solution:

- Let's assume a standard cartesian coordinate system so that we can assign directions to our vectors for momentum.
- Since there are no external forces, we apply conservation of momentum with the initial momenta of m and $3m$ both equal to zero.

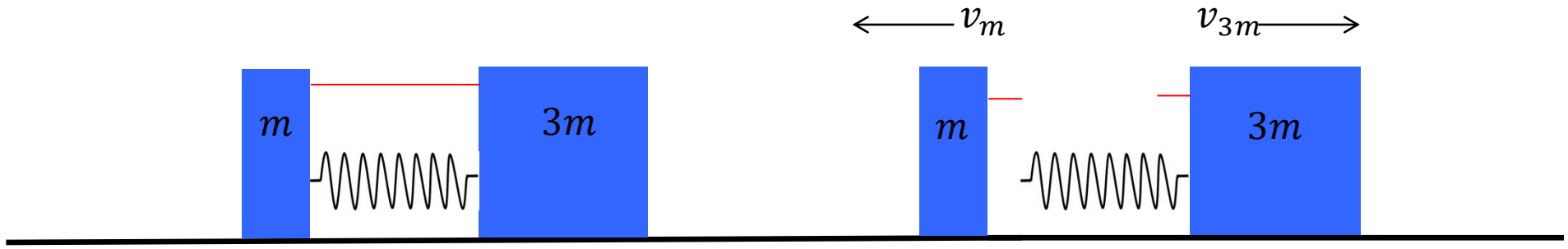
$$0 = \overset{0}{\cancel{p_{im}}} + \overset{0}{\cancel{p_{i3m}}} = p_{fm} + p_{f3m}$$

$$\longrightarrow \quad p_{fm} = -p_{f3m}$$

$$\longrightarrow \quad mv_{fm} = -3mv_{f3m}$$

$$\longrightarrow \quad v_{fm} = -3v_{f3m} = -3\times 2\frac{m}{s} = -6\frac{m}{s}$$

Seems reasonable. The mass is smaller so it should go faster and the sign of the velocity (and momentum) is negative as the mass is moving to the left.

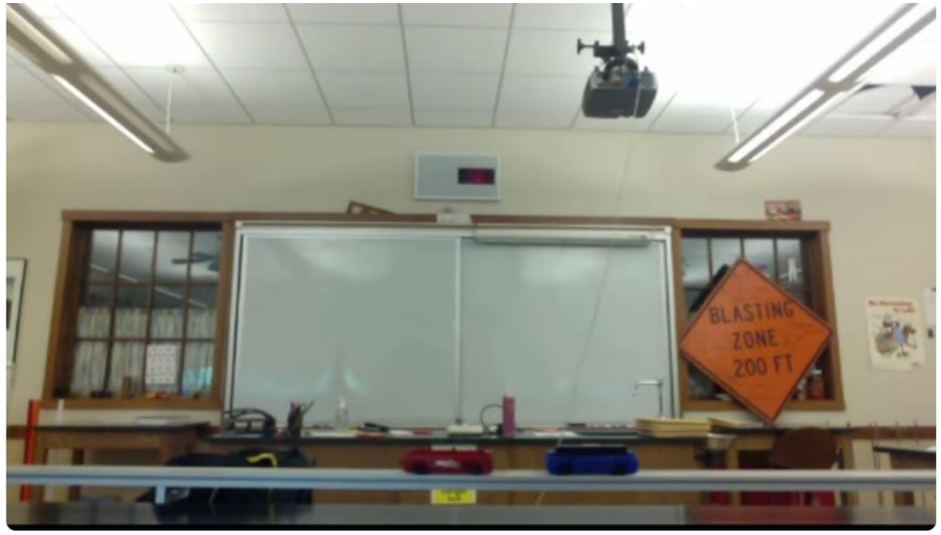


Momentum and Collisions

Example 3:

A cart of mass m_1 moving with an initial velocity $v_{1i} = v$ undergoes a nearly elastic collision in 1D with a second cart of mass m_2 at rest ($v_{2i} = 0$).

- What are the velocities v_{1f} and v_{2f} of the two carts after the collision?
- What are the velocities v_{1f} and v_{2f} of the two carts after the collision if $m_1 = m_2$?
- What are the velocities v_{1f} and v_{2f} of the two carts after the collision if $m_1 \gg m_2$?
- What are the velocities v_{1f} and v_{2f} of the two carts after the collision if $m_1 \ll m_2$?



https://www.youtube.com/watch?v=YhLztPNEcaQ

Momentum and Collisions

- Elastic Collisions

Example 3 Solution:

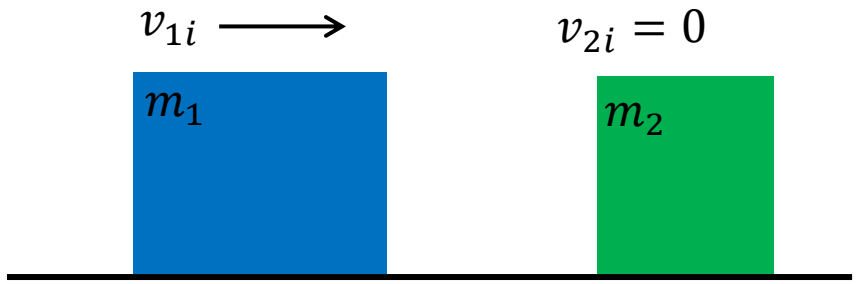
- Since we have a collision, we'll apply conservation of momentum assuming a standard cartesian coordinate system and assume m_1 moves to the right after the collision. Maybe it does, maybe it doesn't. We'll let the math tell us.

$$\Delta p_x = p_{fx} - p_{ix} = 0$$

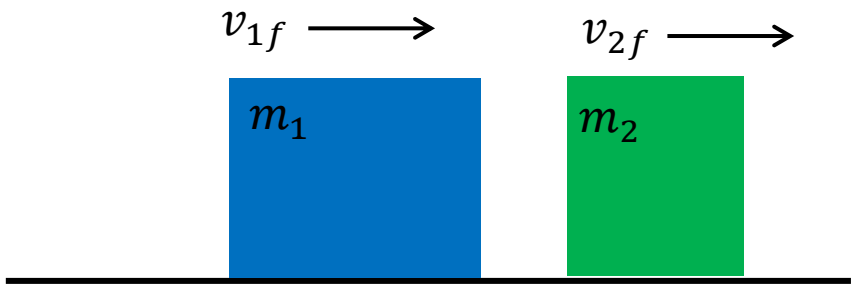
$$p_{ix} = p_{fx} \rightarrow m_1 v_{1i} + m_2 \cancel{v_{2i}}^{0} = m_1 v_{1f} + m_2 v_{2f}$$

$$\longrightarrow \boxed{m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}}$$

- To solve for the final speeds v_{1f} and v_{2f} after the collision, momentum conservation will not give us both. We need another equation. Let's use the fact that the energy of motion is conserved.



Before the collision.



After the collision.

Momentum and Collisions

- Elastic Collisions

Example 3 Solution:

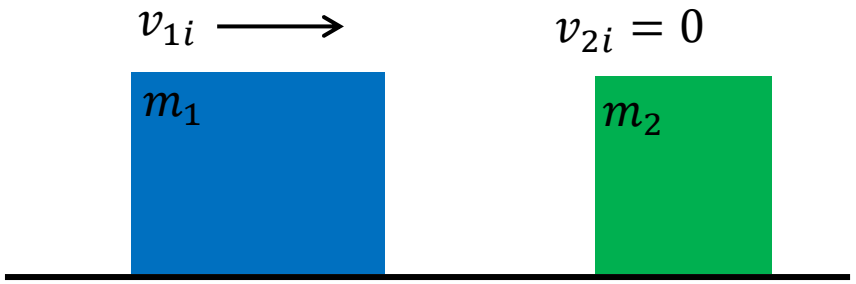
- Conservation of kinetic energy gives us:

$$\Delta K = K_f - K_i = 0 \rightarrow K_f = K_i$$

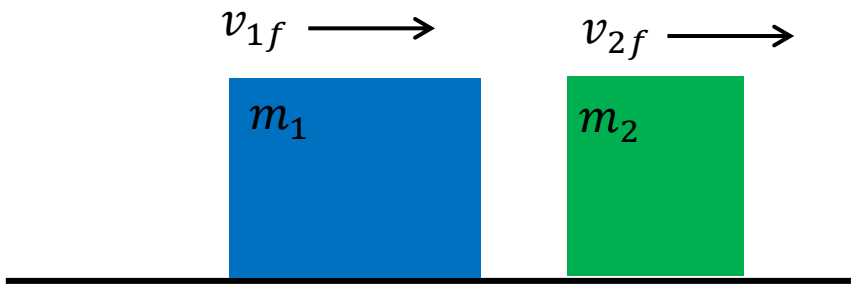
$$\frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 \overset{0}{} = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$$

$$\rightarrow \boxed{\frac{1}{2}m_1 v_{1i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2}$$

- This is our second equation that relates v_{1f} and v_{2f} to each other with the use of conservation of momentum.



Before the collision.



After the collision.

Momentum and Collisions

- Elastic Collisions

Example 3 Solution continued:

- Conservation of Momentum: → $m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}$
- Conservation of Energy: → $\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$

- From here, we have two equations and two unknowns, and we need to find a favorite way to solve this.
- To solve this, I can solve for either v_{1f} or v_{2f} from the equation for conservation of momentum and then substitute the equation I get into the statement of conservation of kinetic energy. I'm going to solve for v_{2f}.

$$v_{2f} = \frac{m_1}{m_2} v_{1i} - \frac{m_1}{m_2} v_{1f} \longrightarrow v_{2f}^2 = \frac{m_1^2}{m_2^2} v_{1i}^2 + \frac{m_1^2}{m_2^2} v_{1f}^2 - 2\frac{m_1^2}{m_2^2} v_{1i} v_{1f}$$

$$m_1 v_{1i}^2 = m_1 v_{1f}^2 + m_2 v_{2f}^2 \longrightarrow m_1 v_{1i}^2 = m_1 v_{1f}^2 + m_2 \left(\frac{m_1^2}{m_2^2} v_{1i}^2 + \frac{m_1^2}{m_2^2} v_{1f}^2 - 2\frac{m_1^2}{m_2^2} v_{1i} v_{1f} \right)$$

Momentum and Collisions

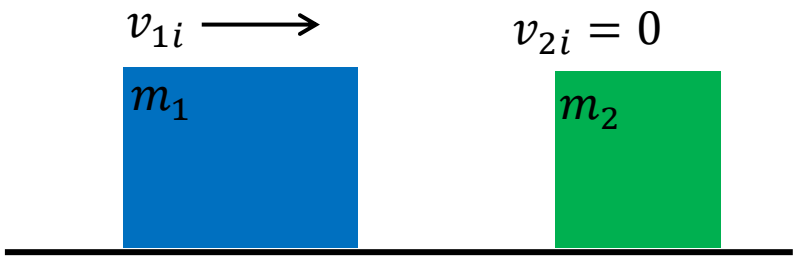
- Elastic Collisions

Example 3 Solution continued:

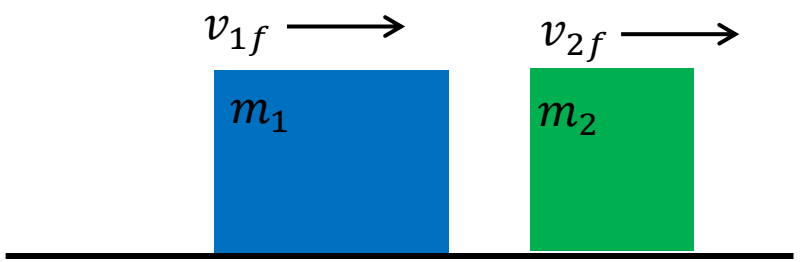
- From here, we have a messy algebra problem, but it is solvable.
- We solve this expression for v_{1f} and substitute the result into our equation for v_{2f} and then solve for v_{2f}.
- I'm not going to do this algebra. Work the solutions out and see if you get the answers below.
- The solutions for v_{1f} and v_{2f} are:

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i}$$



Before the collision.



After the collision.

Momentum and Collisions

- Elastic Collisions

Example 3 Solution continued:

- Are these solutions reasonable? Let's look at a few limiting cases and see.

$m_2 \gg m_1$

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} \sim -\frac{m_2}{m_2} v_{1i} \sim -v_{1i}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} \sim \frac{2m_1}{m_2} v_{1i} \sim 0$$

Here we have maybe a ball (m_1) bouncing off of a solid wall. The wall (m_2) doesn't really move and the ball bounces off with approximately the same speed but in the opposite direction. This was example 1.

$m_1 \gg m_2$

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} \sim \frac{m_1}{m_1} v_{1i} \sim v_{1i}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} \sim \frac{2m_1}{m_1} v_{1i} \sim 2v_{1i}$$

Here we have maybe a semi-truck (m_1) running into the back of a small car (m_2) . The truck hardly slows down but the car gets a big kick and moves forward at a higher speed.

$m_2 = m_1$

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} = 0$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} = v_{1i}$$

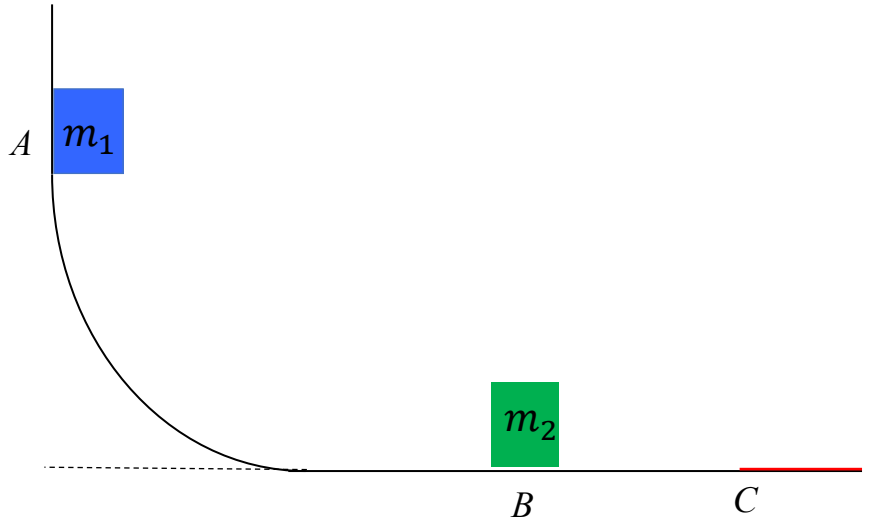
Here, we have maybe, two pool balls striking each other. After the collision m_1 stops moving and m_2 leaves with the speed of m_1.

- In these limiting cases, the solutions seem reasonable.

Momentum and Collisions

Example 4:

Consider the frictionless track on the right where a mass $m_1 = 5kg$ is released from rest at point A. Mass m_1 makes an elastic head-on collision with a mass $m_2 = 10kg$ initially at rest on the horizontal surface.



- What is the maximum height to which mass m_1 rises back up the track after the collision if mass m_1 falls initially through a height of $h = 5m$?

- Suppose that the region past point C, there is friction with coefficient $\mu = 0.9$. After how much distance would it take for mass m_2 to come to rest?

Momentum and Collisions

Example 4 Solution:

- Assuming the system is the mass m_1 and the world, we apply conservation of energy between points A and B to determine the speed of the mass m_1 immediately before the collision.

$$\Delta E = \Delta K_1 + \Delta U_{g1} + \Delta U_s = 0 \quad (\Delta U_s = 0)$$

$$\left(\frac{1}{2}m_1 v_{1f}^2 - \frac{1}{2}m_1 v_{1i}^2\right) + \left(m_1 g y_{1f} - m_1 g y_{1i}\right) = 0 \quad \left(\frac{1}{2}m_1 v_{1i}^2 = 0,\ m_1 g y_{1f} = 0\right)$$

$$v_{1f} = \sqrt{2 g y_{1i}} = \sqrt{2 \times 9.8 \frac{m}{s^2} \times 5m} = 9.9 \frac{m}{s}$$

- At point B, mass m_1 makes an elastic head-on collision with a mass at rest. Applying conservation of momentum and kinetic energy to the collision we can determine expressions for the speeds of and after the collision. The speeds are:

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} = \left(\frac{5kg - 10kg}{5kg + 10kg}\right) \times 9.9 \frac{m}{s} = -3.3 \frac{m}{s}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} = \left(\frac{2 \times 5kg}{5kg + 10kg}\right) \times 9.9 \frac{m}{s} = 6.5 \frac{m}{s}$$

Mass moves m_1 back up the curved portion of the track towards point A while mass m_2 moves to the right toward point C.

Momentum and Collisions

Example 4 Solution continued:

- To determine how far back up the curved portion of the track mass m_1 goes, again we use conservation of energy.

$$\Delta E = \Delta K_1 + \Delta U_{g1} + \Delta U_s = 0 \quad (\Delta U_s = 0)$$

$$\left(\frac{1}{2}m_1 v_{1f}^2 - \frac{1}{2}m_1 v_{1i}^2\right) + \left(m_1 g y_{1f} - m_1 g y_{1i}\right) = 0 \quad (v_{1f} = 0,\ y_{1i} = 0)$$

$$y_{1f} = \frac{v_{1i}^2}{2g} = \frac{\left(-3.3\frac{m}{s}\right)^2}{2\times 9.8\frac{m}{s^2}} = 0.56m$$

- To determine how far mass m_2 goes before coming to rest due to friction past point C, let's use the work-kinetic energy theorem.

$$W = -F_{fr}\Delta x = \Delta K = \frac{1}{2}m_2 v_{2f}^2 - \frac{1}{2}m_2 v_{2i}^2 \quad (v_{2f} = 0)$$

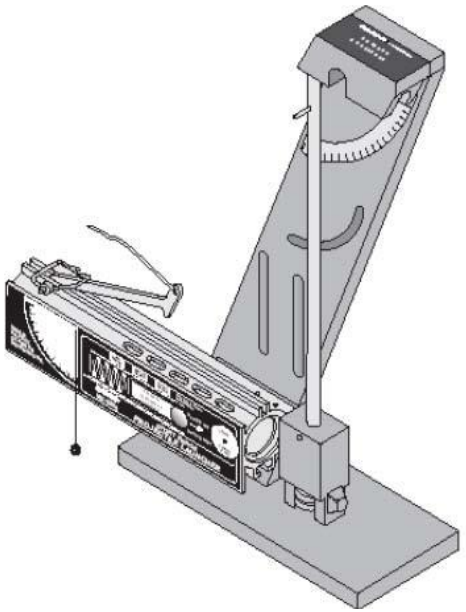
$$\Delta x = \frac{m_2 v_{2i}^2}{2F_{fr}} = \frac{m_2 v_{2i}^2}{2\mu F_N} = \frac{10kg\left(6.5\frac{m}{s}\right)^2}{2\times 0.9\times 10kg\times 9.8\frac{m}{s^2}} = 2.4m$$

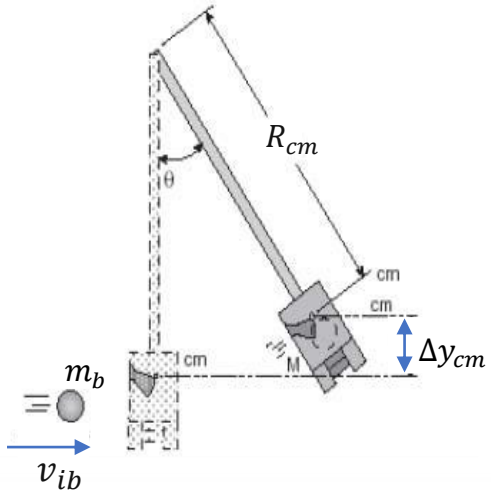
- Note, we could also have used conservation of energy with the change in energy ΔE, of the system (m_2) due to the work done by friction bringing m_2 to rest across the horizontal surface ($\Delta U_{g2} = 0$).

Momentum and Collisions

Example 5:

A ballistic pendulum apparatus is used to study the principles of momentum and energy and to determine the initial velocity of projectiles fired from a launcher. Suppose that a steel ball of mass $m_b = 50g$ launched with an initial velocity v_{ib}. The steel ball undergoes an inelastic collision with a stationary pendulum arm of length $R_{cm} = 30.5cm$ and mass $M = 250g$. After the collision, the center of mass of the ball and pendulum arm rises from its lowest point thought a height Δy_{cm}, where it momentarily comes to rest at an angle $\theta = 27^0$measured with respect to the vertical.



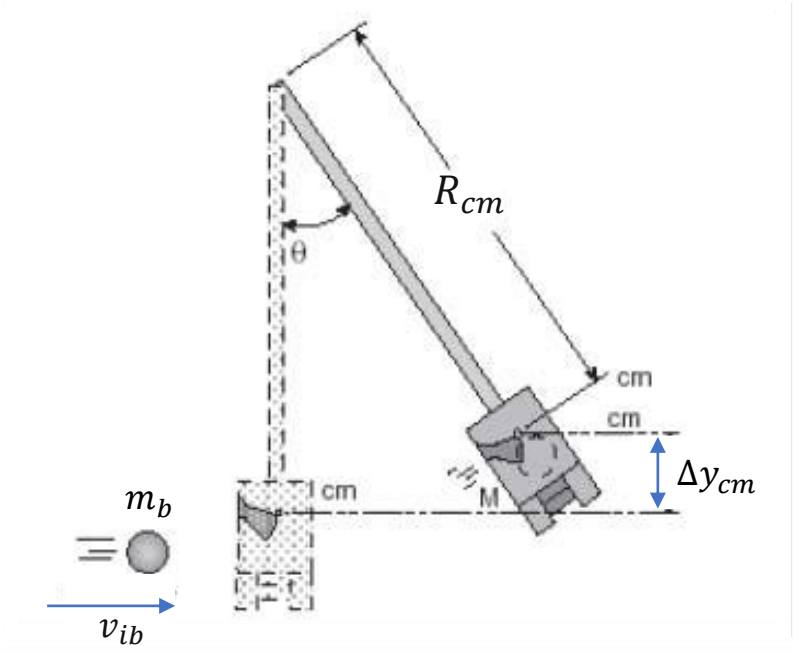


- What is the speed of the ball and pendulum immediately after the collision?
- With what speed, v_{ib}, was the ball launched into the pendulum arm?
- What fraction of the ball's initial kinetic energy was lost in the collision?
- If the spring in the launcher were initially compressed by $9cm$, what what is the stiffness of the spring?

Momentum and Collisions

Example 5 Solution:

- Let's assume that the collision is inelastic since the two objects stick together after the collision.
- To see if the collision is actually inelastic or not, we'll calculate the difference in kinetic energy before and after the collision. If the difference is not zero, then the collision is inelastic.
- To calculate the initial velocity of the ball, we use conservation of momentum and let V be the speed of the ball and pendulum arm after the collision.



$$\Delta p_x = p_{fx} - p_{ix} = 0$$

$$p_{ix} = p_{fx} \rightarrow m_b v_{ib} + m_P \overset{0}{\cancel{v_{iP}}} = m_b v_{bf} + m_P v_{Pf}$$

$$m_b v_{ib} = (m_b + M)V$$

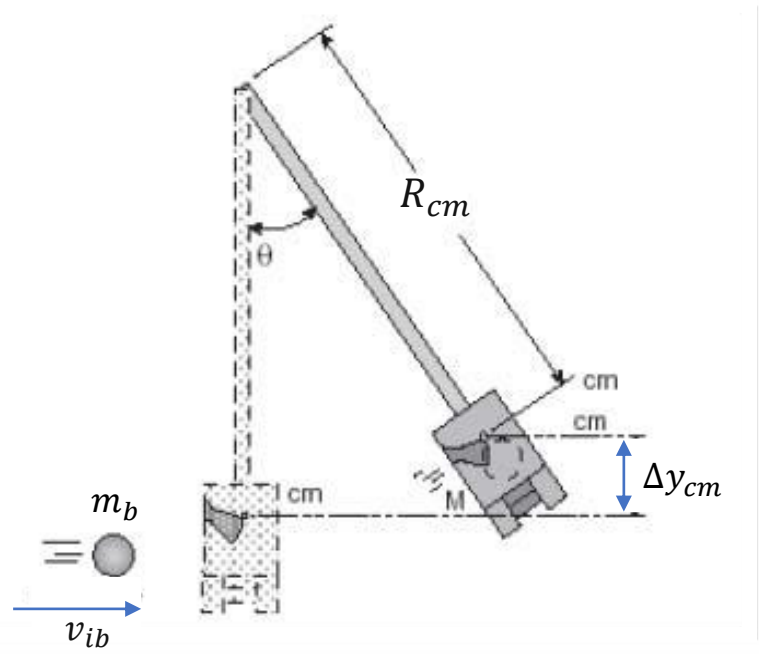
$$\rightarrow \boxed{v_{ib} = \frac{(m_b + M)}{m_b} V}$$

In order to determine the launch speed (v_{ib}) of the ball before the collision, we need to know the speed of the ball and pendulum (V) after the collision.

Momentum and Collisions

Example 5 Solution continued:

- The collision itself does not provide any more information about the launch speed of the ball nor does it provide us with a means to determine the speed of the ball and pendulum after the collision.
- Since we do not know directly if the collision is elastic or inelastic, we can't simply apply conservation of kinetic energy.
- What we can do however is watch how high the ball and pendulum arm rise in the air after the collision and apply conservation of energy after the collision.
- After the collision the initial kinetic energy of the ball and pendulum arm are converted to gravitational potential energy when the ball and pendulum arm come to rest.



$$\Delta E = \Delta K + \Delta U_g + \Delta U_s = 0$$

($\Delta U_s = 0$)

$$\Delta E = \left[\frac{1}{2}(m_b + M)v_f^2 - \frac{1}{2}(m_b + M)v_i^2\right] + \left[(m_b + M)gy_f - (m_b + M)gy_i\right] = 0$$

($v_f = 0$, $y_i = 0$)

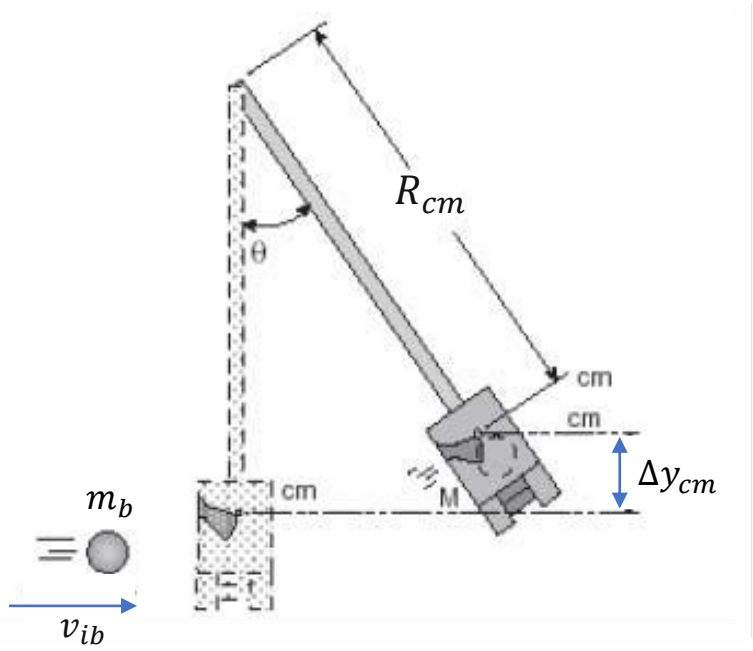
The ball and pendulum arm lose kinetic energy as the system rises to its final height.

The ball and pendulum arm gain gravitational potential energy as the system rises to its final height.

Momentum and Collisions

Example 5 Solution continued:

- We need to determine the height that the center-of-mass rises through.
- We're not going to talk about the center-of-mass, but the idea is that we have an extended object (the pendulum arm) and we'd like to keep using our point-like approximation even though we have an extended object.
- To keep the point-like approximation we assume that all of the mass of the object can be placed at the center-of-mass of the object.
- Normally this is at the center of a uniform distribution of mass, like the ball. Its center-of-mass is at its center.
- To determine the center-of-mass of an extended object we see where it balances, on say your finger. Consider any standard ruler. Where is its center-of-mass? Lay it across your finger and where it balances is the center-of-mass, meaning that the mass on one side of your finger is the same as the mass on the other side of your finger, which should be equidistant from both ends.
- Here, we'd lay the ball and pendulum arm across your finger and where it balances, like the ruler, is the location of the center-of-mass.



Momentum and Collisions

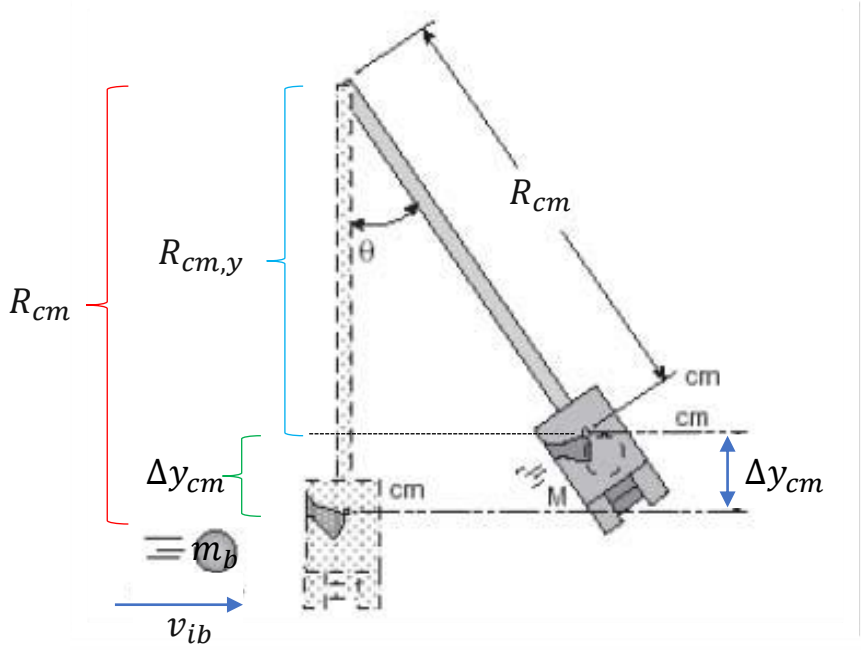
Example 5 Solution continued:

- We need to determine the height that the center-of-mass of the ball and pendulum arm rise through, Δy_{cm}.

- We do this using the geometry in the problem and the fact that the length R_{cm} of the center-of-mass of the ball and pendulum arm is a constant.

$$R_{cm} = R_{cm,y} + \Delta y_{cm} \rightarrow \Delta y_{cm} = R_{cm} - R_{cm,y}$$

$$\rightarrow \Delta y_{cm} = R_{cm} - R_{cm,y} = R_{cm} - R_{cm}\cos\theta$$



- Returning to our statement of conservation of energy:

$$\Delta E = \left[\frac{1}{2}(m_b + M)\overbrace{v_f^2}^{0} - \frac{1}{2}(m_b + M)v_i^2\right] + \left[(m_b + M)gy_f - (m_b + M)\overbrace{gy_i}^{0}\right] = 0$$

$$-\frac{1}{2}(m_b + M)V^2 + (m_b + M)gy_f = -\frac{1}{2}(m_b + M)V^2 + (m_b + M)g(R_{cm} - R_{cm}\cos\theta) = 0$$

$$\boxed{V = \sqrt{2gR_{cm}(1 - \cos\theta)}}$$

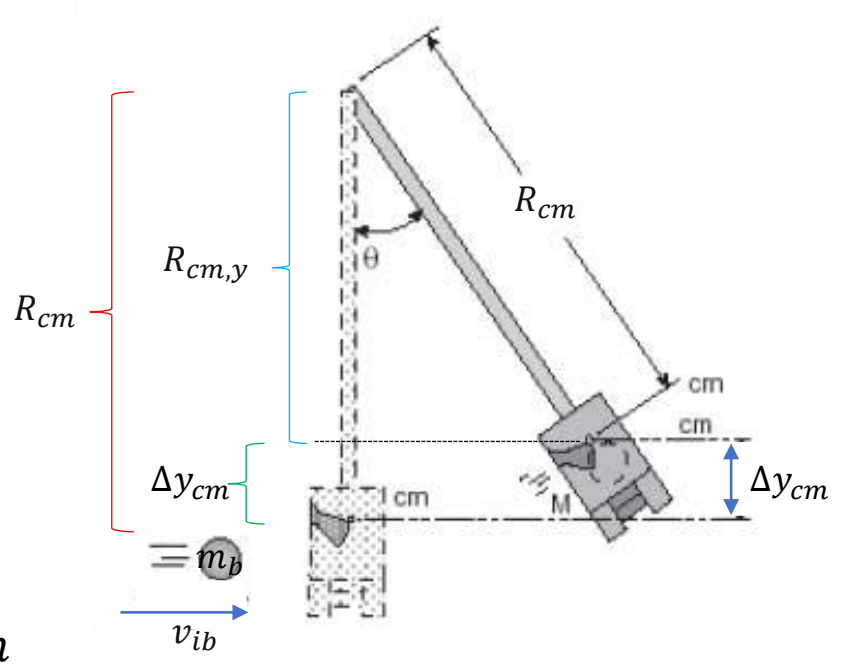
Momentum and Collisions

Example 5 Solution continued:

- Combining the expression for the speed of the ball and pendulum arm (V) with the expression for the initial velocity of the ball (v_{ib}) we can determine the value of the launch speed of the ball.

$$v_{ib} = \frac{(m_b + M)}{m_b} V = \frac{(m_b + M)}{m_b} \sqrt{2gR_{cm}(1 - \cos\theta)}$$

$$v_{ib} = \frac{(0.050kg + 0.250kg)}{0.050kg} \sqrt{2 \times 9.8 \frac{m}{s^2} \times 0.305m \times (1 - \cos 27)} = 4.8 \frac{m}{s}$$



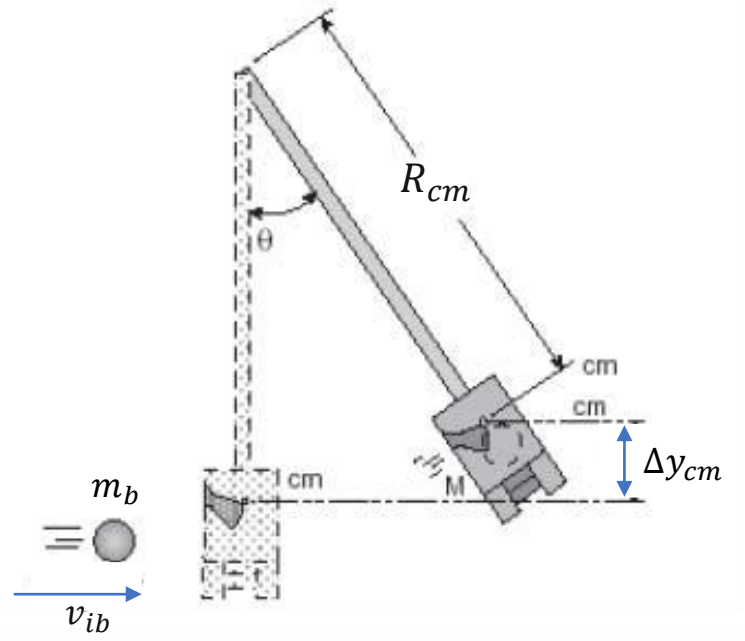
- Does this seem like a reasonable launch speed? That depends on what are we comparing it to?

- These are the same launchers you "used" a few weeks ago in lab to determine the launch speed of a projectile. If steel balls were used, then how does this launch speed compare to what you determined in the video of the projectile experiment? The launch speeds that students typically get are between $4.5\frac{m}{s}$ and $5.0\frac{m}{s}$. So probably reasonable.

Momentum and Collisions

Example 5 Solution continued:

- To determine if the collision is inelastic or elastic, let's calculate the change in kinetic energy before and after the collision.



$$K_i = \frac{1}{2}m_b v_{ib}^2 = \frac{1}{2}\times 0.050kg \times \left(4.8\frac{m}{s}\right)^2 = 0.586J$$

$$K_f = \tfrac{1}{2}(m_b + M)V^2 = \tfrac{1}{2}(m_b + M)\left(2gR_{cm}(1-\cos\theta)\right) = (m_b + M)\, gR_{cm}(1-\cos\theta)$$

$$K_f = (0.050kg + 0.250kg)9.8\frac{m}{s^2}\times 0.305m(1-\cos 27) = 0.099J$$

$\Delta K = K_f - K_i = 0.099J - 0.586J = -0.487J$ Since $\Delta K \neq 0$, this is an inelastic collision.

- The percent of the initial kinetic energy lost in the collision is:

$$\%lost = \frac{\Delta K}{K_i}\times 100\% = \frac{-0.487J}{.586J}\times 100\% = -83\%$$

83% of the initial kinetic energy is lost (mainly to sound) in the collision.

Momentum and Collisions

Example 5 Solution continued:

- One last thing before we move on.

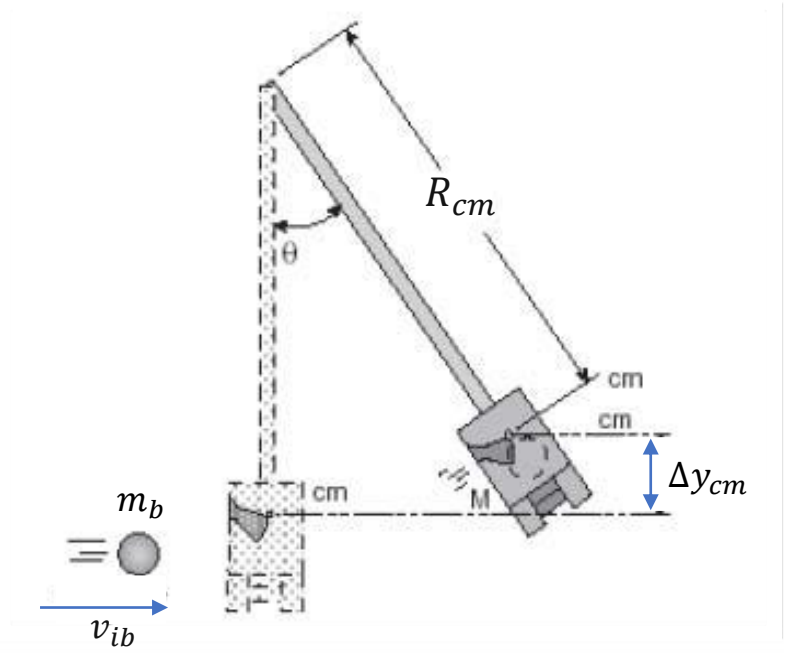
- We calculated the change in energy using actual numbers. What if we didn't have numbers in the problem? Could we still calculate a result for the loss of kinetic energy? The answer, of course, is yes.

- The fraction f of the initial kinetic energy lost in this collision is given by:

$$f = \frac{\Delta K}{K_i} = \frac{K_f - K_i}{K_i} = \frac{K_f}{K_i} - 1 = \frac{\frac{1}{2}(m_b + M)V^2}{\frac{1}{2}m_b v_{ib}^2} - 1$$

- We can write the final speed of the ball and pendulum arm after the collision (V) in terms of the initial speed of the ball before the collision (v_{ib}) using conservation of momentum.

$$V = \frac{m_b}{(m_b + M)} v_{ib} \rightarrow V^2 = \left(\frac{m_b}{m_b + M}\right)^2 v_{ib}^2$$



Momentum and Collisions

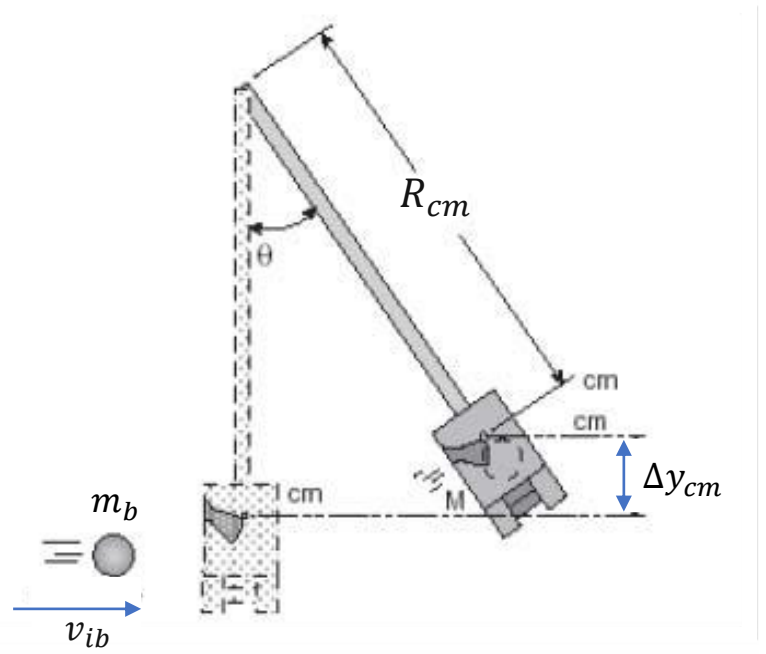
Example 5 Solution continued:

- The fraction f (and if I multiply the result by 100%, the percent of the kinetic energy lost) is:

$$f = \frac{\frac{1}{2}(m_b + M)V^2}{\frac{1}{2}m_b v_{ib}^2} - 1 = \frac{(m_b + M)\left(\frac{m_b}{m_b + M}\right)^2 v_{ib}^2}{m_b v_{ib}^2} - 1$$

$$\rightarrow \boxed{f = \frac{m_b}{m_b + M} - 1}$$

The fraction of the initial kinetic energy lost in the collision depends on the ratio of the masses and not any other parameters in the experiment.



- For the numbers we are given for the mass of the ball and the mass of the pendulum arm the fraction of the initial kinetic energy lost in the collision is:

$$f = \frac{m_b}{m_b + M} - 1 = \frac{0.05kg}{0.05kg + 0.250kg} - 1 = -0.83 \rightarrow \%lost = -83\%$$

We lose 83% of the initial kinetic energy in the collision. This is the same value we obtained by direct calculation of the difference in kinetic energy before and after the collision.

Momentum and Collisions

Example 6:

In the basement of the Integrated Science and Engineering Complex (ISEC) the department of physics & astronomy has a small particle accelerator. We use this accelerator to routinely accelerate alpha particles into targets of composed of various elements in order to determine the elemental makeup of the target. In a famous experiment called Rutherford's experiment, we fire alpha particles at a target composed of gold atoms. An alpha particle (a helium nucleus) is accelerated to a certain speed and most of the time, the alpha particles pass right through the gold foil target never having interacted with anything. A small fraction of the time though, some alpha particles makes an elastic head-on collision with a stationary gold nucleus in the target.

- Suppose that an alpha particle starts from rest and experiences a constant force of $F = 1.6\times10^{-13}N$ over $\Delta x = 3.3m$, what is the speed of the alpha particle?

- What are the speeds of the alpha particle and the gold nucleus after they collide?

- What percentage of the original kinetic energy of the alpha particle is transferred to the gold nucleus?

Momentum and Collisions

Example 6 Solution:

Applying conservation of momentum we have:

$$\Delta p_x = p_{fx} - p_{ix} = 0$$

$$p_{ix} = p_{fx} \rightarrow m_\alpha v_{i\alpha} + m_{Au}\overset{0}{\cancel{v_{iAu}}} = m_\alpha v_{f\alpha} + m_{Au} v_{fAu}$$

$$\rightarrow \quad \boxed{m_\alpha v_{i\alpha} = m_\alpha v_{f\alpha} + m_{Au} v_{fAu}}$$

Applying conservation of kinetic energy we have:

$$\Delta K = K_f - K_i = 0 \rightarrow K_f = K_i$$

$$\frac{1}{2}m_\alpha v_{i\alpha}^2 + \frac{1}{2}m_{Au}\overset{0}{\cancel{v_{iAu}^2}} = \frac{1}{2}m_\alpha v_{f\alpha}^2 + \frac{1}{2}m_{Au} v_{fAu}^2$$

$$\rightarrow \quad \boxed{\frac{1}{2}m_1 v_{1i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2}$$

Solving these simultaneously for the final speeds of the alpha particle and the gold nucleus we get:

$$\boxed{v_{f\alpha} = \left(\frac{m_\alpha - m_{Au}}{m_\alpha + m_{Au}}\right) v_{i\alpha}}$$

$$\boxed{v_{fAu} = \left(\frac{2m_\alpha}{m_\alpha + m_{Au}}\right) v_{i\alpha}}$$

Momentum and Collisions

Example 6 Solution:

To determine the final speeds of the alpha particle and the gold nucleus, we need to know the initial speed. We'll assume that the alpha particle's mass is approximately four times the mass of a proton.

From the work-kinetic energy theorem we have:

$$W = F \cdot \Delta x \cdot \cos\theta = F \cdot x \cdot \cos\theta = Fx = \Delta K = \frac{1}{2}m_\alpha v_{f\alpha}^2 - \frac{1}{2}m_\alpha \overset{0}{v_{i\alpha}^2} = \frac{1}{2}m_\alpha v_{f\alpha}^2$$

$$v_{f\alpha} = \sqrt{\frac{2Fx}{m_\alpha}} = \sqrt{\frac{12Fx}{4 \cdot m_p}} = \sqrt{\frac{2 \cdot 1.6\times10^{-13}N \cdot 3.3m}{4 \cdot 1.67\times10^{-27}kg}} = 1.26\times10^7\frac{m}{s}$$

The final speeds of the alpha particle and the gold nucleus after the collision are given from (where we'll write the masses of the alpha particle and gold nucleus in atomic (or unified) mass units as $4amu$ and $197amu$ respectively.

$$v_{f\alpha} = \left(\frac{m_\alpha - m_{Au}}{m_\alpha + m_{Au}}\right)v_{i\alpha} = \left(\frac{4amu - 197amu}{4amu + 197amu}\right) \cdot 1.27\times10^6\frac{m}{s} = -1.22\times10^6\frac{m}{s}$$

$$v_{fAu} = \left(\frac{2m_\alpha}{m_\alpha + m_{Au}}\right)v_{i\alpha} = \left(\frac{2 \cdot 4amu}{4amu + 197amu}\right) \cdot 1.27\times10^6\frac{m}{s} = 5.1\times10^4\frac{m}{s}$$

Momentum and Collisions

Example 6 Solution:

The fraction of the energy transferred to the gold nucleus:

$$\Delta E = 0 \rightarrow E_i = E_f \rightarrow K_{\alpha i} = K_{\alpha f} + K_{Auf} \rightarrow K_{Auf} = K_{i\alpha} - K_{f\alpha}$$

$$f = \frac{K_{fAu}}{K_{i\alpha}} = \frac{K_{i\alpha} - K_{f\alpha}}{K_{i\alpha}} = \frac{\frac{1}{2}m_\alpha v_{i\alpha}^2 - \frac{1}{2}m_\alpha v_{f\alpha}^2}{\frac{1}{2}m_\alpha v_{i\alpha}^2} = 1 - \frac{v_{f\alpha}^2}{v_{i\alpha}^2} = 1 - \frac{\left(\frac{m_\alpha - m_{Au}}{m_\alpha + m_{Au}}\right)^2 v_{i\alpha}^2}{v_{i\alpha}^2} = 1 - \left(\frac{m_\alpha - m_{Au}}{m_\alpha + m_{Au}}\right)^2$$

$$f = 1 - \left(\frac{4amu - 197amu}{4amu + 197amu}\right)^2 = 0.078$$

Or about 7.8% of the alpha particles initial energy gets transferred to the gold nucleus.

Momentum and Collisions

- Inelastic Collisions in 2D

Example 7:

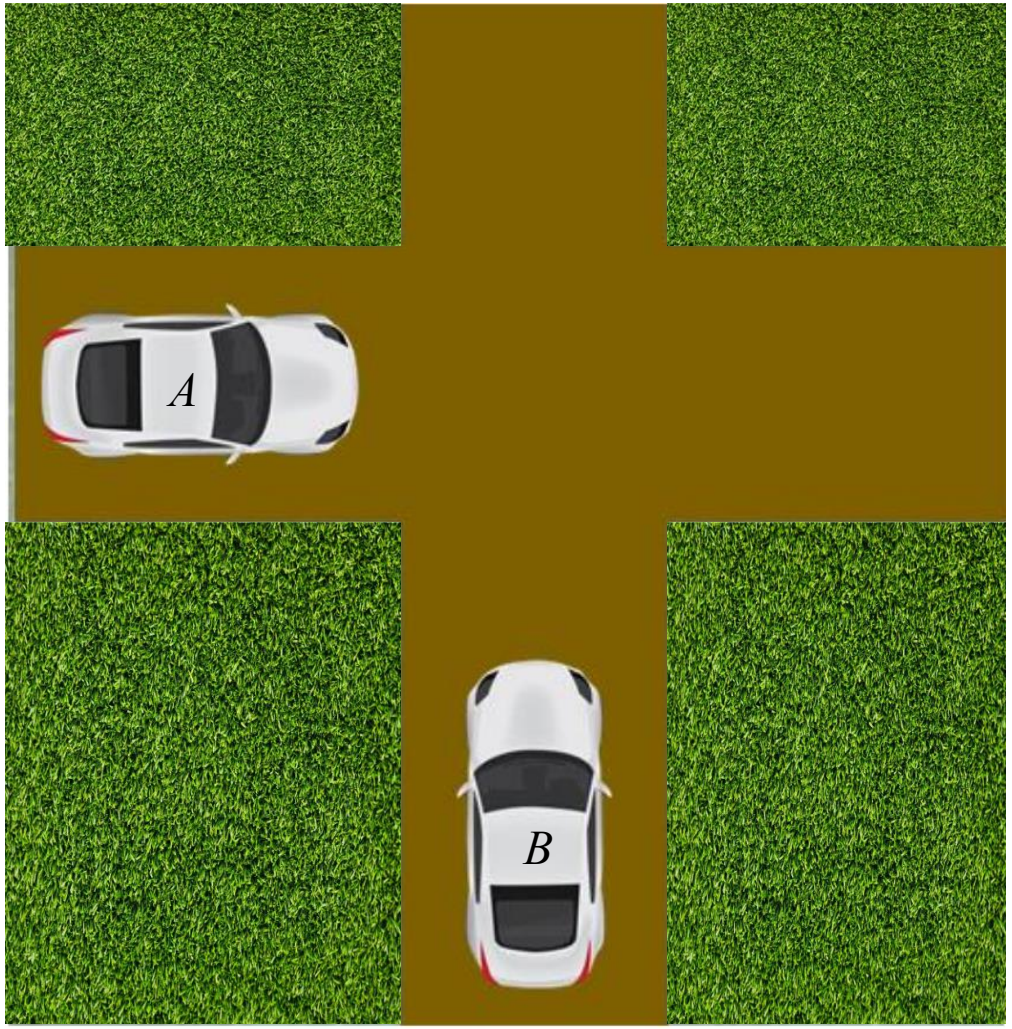
Consider two cars approaching an intersection from two different directions.

Let the car A be traveling from the west towards the east when it approaches a 4-way intersection while car B is traveling from the south towards the north when it approaches the same 4-way intersection.

For some reason, neither car stops at the intersection and the two cars crash and move off together with a common final speed V at an angle θ measured counterclockwise with respect to east.

Let the mass of the car A and and car B be $m_{car,A} = 1500kg$ and $m_{car,B} = 2500kg$, respectively.

If car A's velocity is $v_{ix,A} = 25\frac{m}{s}$ before the collision and the velocity of car B is $v_{ix,B} = 20\frac{m}{s}$ before the collision, what is the magnitude of their velocity after impact, V, and at what angle θ does the crash make with respect to east?



Momentum and Collisions

Example 7 Solution:

- Assuming a standard cartesian coordinate system, we have to conserve momentum in both the horizontal and vertical directions.

- In the horizontal direction:

$$\Delta p_x = p_{fx} - p_{ix} = 0$$

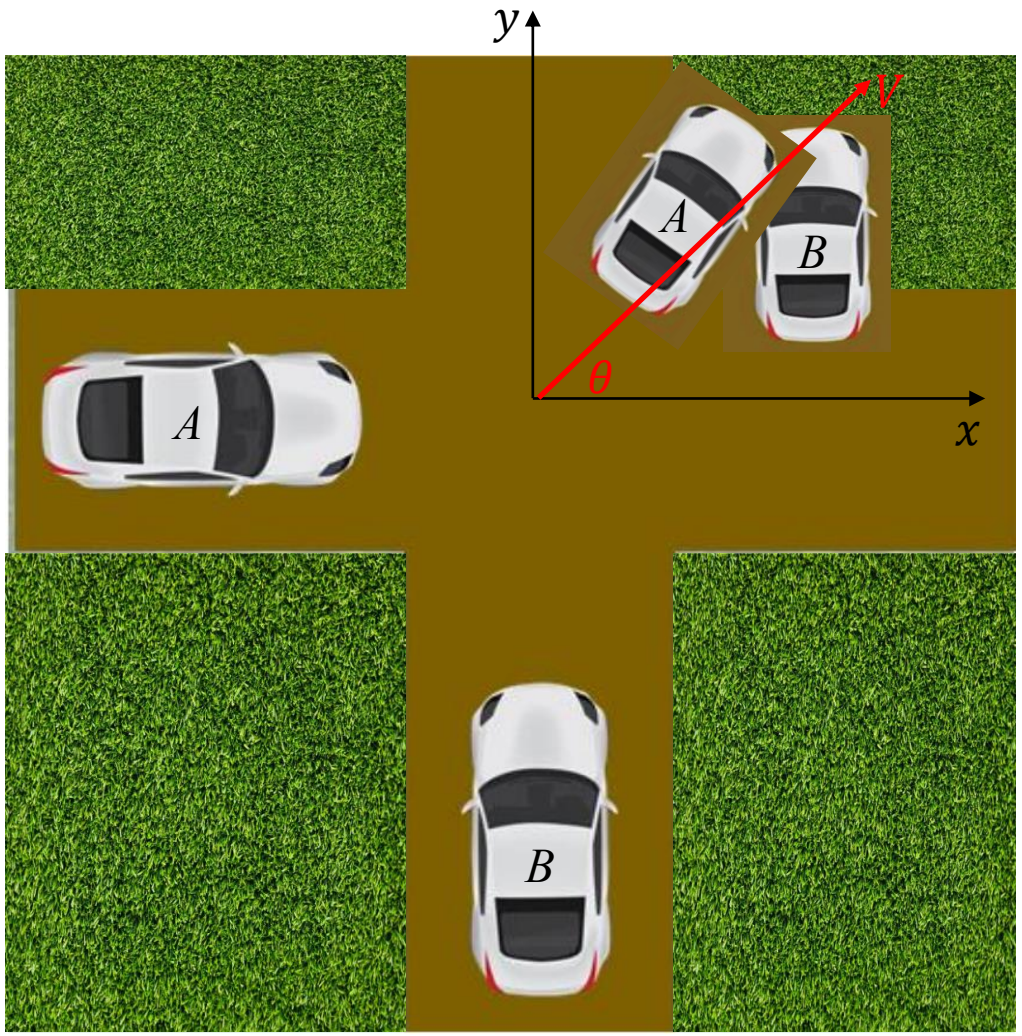
$$p_{ix} = p_{fx} \rightarrow m_A v_{iA} = (m_A + m_B)V_{fx} = (m_A + m_B)V\cos\theta$$

- In the vertical direction:

$$\Delta p_y = p_{fy} - p_{iy} = 0$$

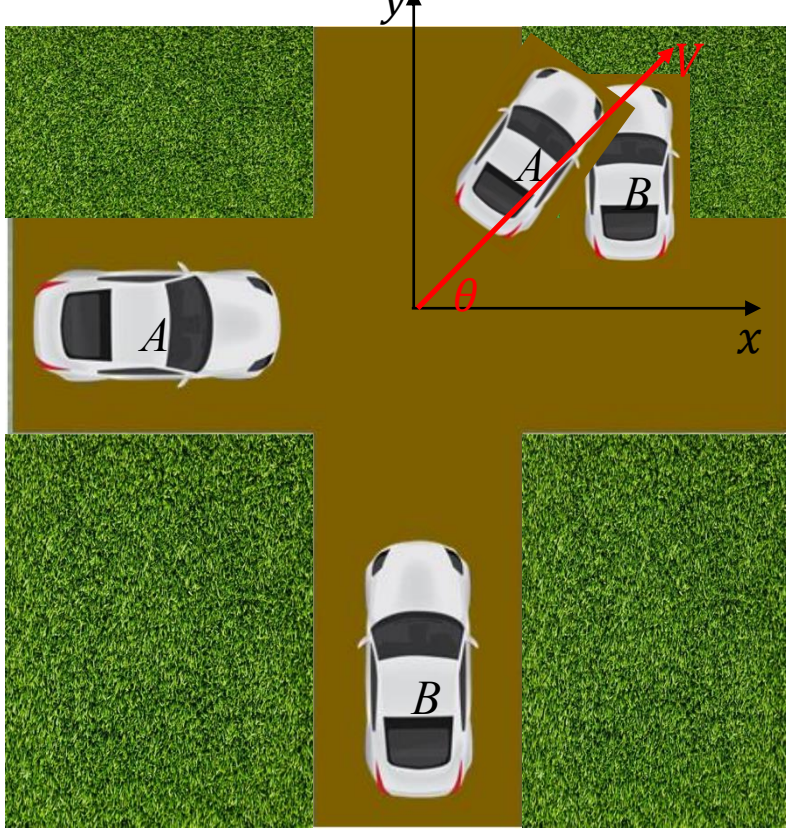
$$p_{iy} = p_{fy} \rightarrow m_B v_{iB} = (m_A + m_B)V_{fy} = (m_A + m_B)V\sin\theta$$

- The easiest way to solve this is to divide the two expressions (or solve one for V and substitute into the other) and solve for the angle θ.



Momentum and Collisions

Example 7 Solution:



- Dividing the two expressions we get:

$$\frac{(m_A + m_B)V\sin\theta}{(m_A + m_B)V\cos\theta} = \frac{m_B v_{iB}}{m_A v_{iA}} \rightarrow \tan\theta = \frac{m_B v_{iB}}{m_A v_{iA}} = \frac{2500kg \times 20\frac{m}{s}}{1500kg \times 25\frac{m}{s}} = 1.33$$

$$\longrightarrow \quad \theta = \tan^{-1} 1.33 = 53.1^0$$

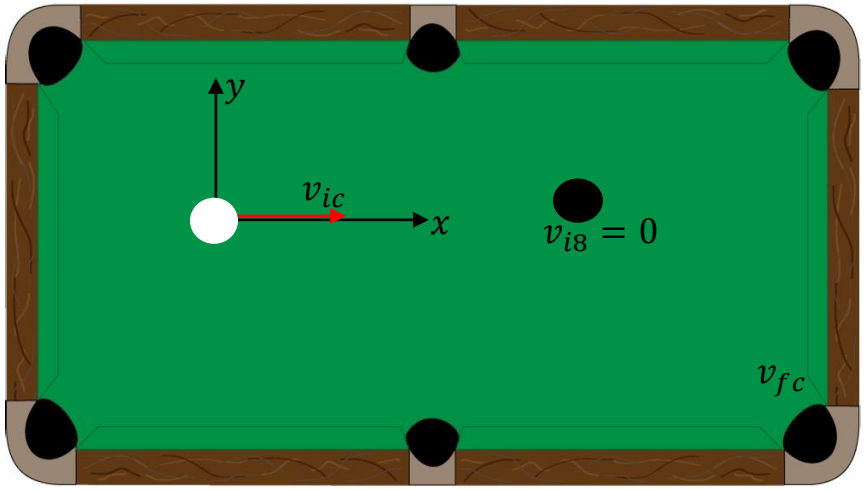
- Then we can use either of our previous two expressions to determine the speed of the cars after the collision:

$$m_A v_{iA} = (m_A + m_B)V\cos\theta \rightarrow V = \frac{m_A v_{iA}}{(m_A + m_B)\cos\theta} = \frac{1500kg \times 25\frac{m}{s}}{(1500kg + 2500kg)\cos 53.1} = 15.6\frac{m}{s}$$

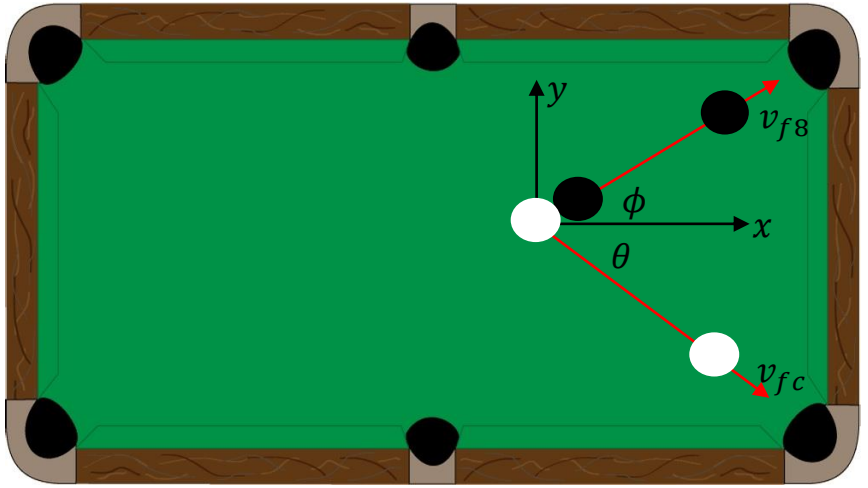
$$m_B v_{iB} = (m_A + m_B)V\sin\theta \rightarrow V = \frac{m_B v_{iB}}{(m_A + m_B)\sin\theta} = \frac{2500kg \times 20\frac{m}{s}}{(1500kg + 2500kg)\sin 53.1} = 15.6\frac{m}{s}$$

Momentum and Collisions

- Elastic Collisions in 2D



Before the collision.



After the collision.

Example 8:

In the game of pool, 15 balls are used along with one cue ball. The cue ball is used to hit the other 15 balls into one of the pockets. On average, the balls are 2.25 inches in diameter and all balls weigh 5.5 ounces ($m_{ball} = 0.15kg$) except or the cue, which weighs slightly at 6 ounces ($m_{cue} = 0.26kg$). The cue ball weighs slightly more so we can separate it from the other balls when they enter a pocket.

Consider two balls (the cue ball and the 8-ball) each having the same mass $m = 0.15kg$.

Assume that the cue ball is initially traveling with an initial speed of $v_{i,c} = 3\frac{m}{s}$ horizontally across the table when it makes a glancing collision with the 8-ball, initially at rest $v_{i,8} = 0\frac{m}{s}$.

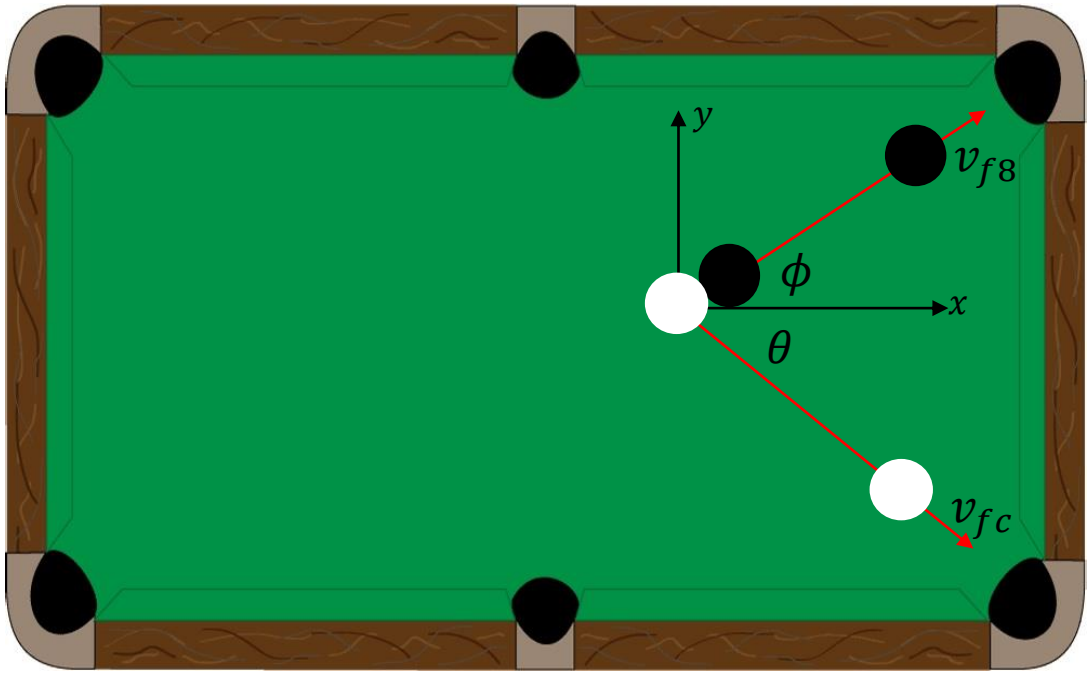
Let the cue ball scatter (be deflected from the horizontal) by an angle of θ, while the 8-ball scatters (is deflected from the horizontal) by and angle ϕ.

What are the speeds of the cue ball (v_{fc}) and the 8-ball (v_{f8}) after the collision?

Momentum and Collisions

Example 8 Solution:

In order to determine the speeds of the cue ball (v_{fc}) and the 8-ball (v_{f8}) after the collision, we'll apply conservation of momentum both horizontally and vertically and assume that the collision is elastic (no losses to sound – probably not realistic) so that we can apply conservation of kinetic energy too.



- Conservation of momentum in the horizontal direction:

$$\Delta p_x = p_{fx} - p_{ix} = 0$$

$$p_{ix} = p_{fx} \rightarrow m_c v_{ic} = m_c v_{fcx} + m_8 v_{f8x}$$

$$\boxed{m_c v_{ic} = m_c v_{fc} \cos\theta + m_8 v_{f8} \cos\phi}$$

- Conservation of momentum in the vertical direction:

$$\Delta p_y = p_{fy} - p_{iy} = 0$$

$$p_{iy} = p_{fy} \rightarrow 0 = m_c v_{fcy} + m_8 v_{f8y}$$

$$\boxed{0 = -m_c v_{fc} \sin\theta + m_8 v_{f8} \sin\phi}$$

- Conservation of kinetic energy:

$$\Delta K = K_f - K_i = 0$$

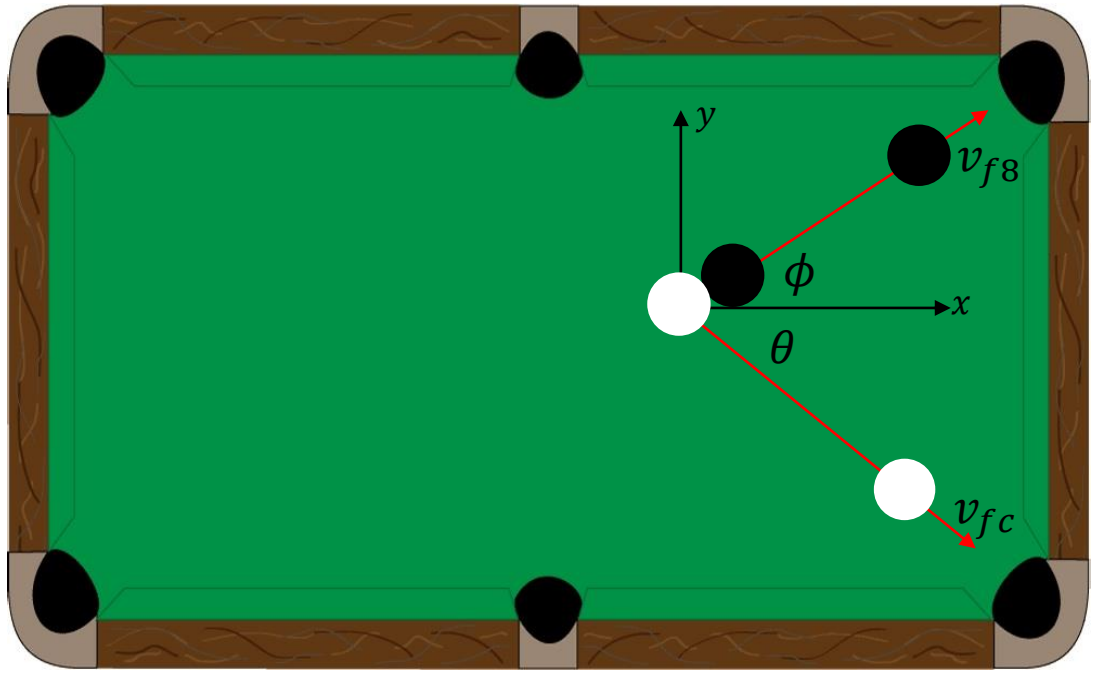
$$K_f = K_i \rightarrow \frac{1}{2} m_c v_{ic}^2 = \frac{1}{2} m_c v_{fc}^2 + \frac{1}{2} m_8 v_{f8}^2$$

$$\boxed{\frac{1}{2} m_c v_{ic}^2 = \frac{1}{2} m_c v_{fc}^2 + \frac{1}{2} m_8 v_{f8}^2}$$

Momentum and Collisions

Example 8 Solution continued:

- The last two equations have 4 unknowns, v_{fc}, v_{f8}, θ, and ϕ.
- Unfortunately, we only have (and only can generate) 3 equations.
- We can solve for at most, 3 of these unknowns.
- In scattering problems such as these, we usually specify a location for a detector to detect one of the scattered particles.
- If instead of pool balls we had say a helium atom incident on a gold atom (as in the last lecture) and the helium atom scattered from the gold as in this example, we'd have to put a detector somewhere to detect either the scattered helium or the scattered gold.
- This means that for this problem with the pool balls, we need to specify one of the angles in the problem. Suppose that we know angle θ and the detector here would be the pocket that the cue ball goes into, unfortunately causing you to lose the game.



Momentum and Collisions

Example 8 Solution continued:

- We have our equations for conservation of momentum horizontally and vertically in general (for any two masses):

$$m_c v_{ic} = m_c v_{fc} \cos\theta + m_8 v_{f8} \cos\phi$$

$$0 = -m_c v_{fc} \sin\theta + m_8 v_{f8} \sin\phi$$

- And for conservation of kinetic energy, in general (for any two masses):

$$\frac{1}{2} m_c v_{ic}^2 = \frac{1}{2} m_c v_{fc}^2 + \frac{1}{2} m_8 v_{f8}^2$$

- To make this problem easier (as we have here) is to make the masses equal (or a multiple of each other) so that we divide out the mass terms.

- Conservation of momentum and kinetic energy become:

$$v_{ic} = v_{fc} \cos\theta + v_{f8} \cos\phi$$

$$0 = -v_{fc} \sin\theta + v_{f8} \sin\phi$$

$$v_{ic}^2 = v_{fc}^2 + v_{f8}^2$$

Momentum and Collisions

Example 8 Solution continued:

- Here, find your favorite way to solve this system of equations.
- We have three unknowns, v_{fc}, v_{f8}, and ϕ.
- My solution method is to solve for the unknown angle first by squaring the two equations for momentum.

$$v_{ic} = v_{fc}\cos\theta + v_{f8}\cos\phi \longrightarrow v_{ic}^2 = v_{fc}^2\cos^2\theta + v_{f8}^2\cos^2\phi + 2v_{fc}v_{f8}\cos\theta\cos\phi$$

$$0 = -v_{fc}\sin\theta + v_{f8}\sin\phi \longrightarrow 0 = v_{fc}^2\sin^2\theta + v_{f8}^2\sin^2\phi - 2v_{fc}v_{f8}\sin\theta\sin\phi$$

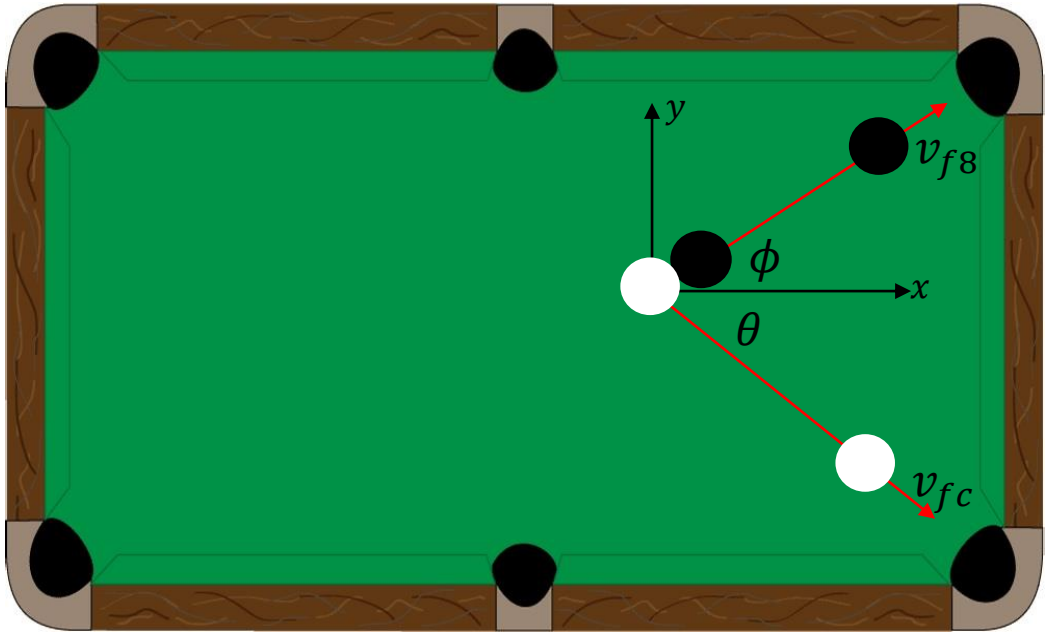
- Let's add these two equations together so that we can 1. use $\sin^2\alpha + \cos^2\alpha = 1$, and 2. use our equation of kinetic energy.
- Adding these two equations together and substituting the equation for kinetic energy:

$$v_{ic}^2 = v_{fc}^2(\cos^2\theta + \sin^2\theta) + v_{f8}^2(\cos^2\phi + \sin^2\phi) + 2v_{fc}v_{f8}(\cos\theta\cos\phi - \sin\theta\sin\phi)$$

$$\longrightarrow v_{ic}^2 = \underbrace{v_{fc}^2 + v_{f8}^2}_{\text{Conservation of kinetic energy}} = \underbrace{v_{fc}^2 + v_{f8}^2 + 2v_{fc}v_{f8}(\cos\theta\cos\phi - \sin\theta\sin\phi)}_{\text{Adding together the square of the momenta}}$$

Momentum and Collisions

Example 8 Solution continued:



- Subtracting out the common terms we get:

$$0 = 2v_{fc}v_{f8}(\cos\theta\cos\phi - \sin\theta\sin\phi)$$

- Assuming that $v_{fc} \neq 0$ and $v_{f8} \neq 0$, we get

$$0 = \cos\theta\cos\phi - \sin\theta\sin\phi$$

- This relates the one unknown angle ϕ to the known angle θ using a trigonometric identity.

$$0 = \cos\theta\cos\phi - \sin\theta\sin\phi = \cos(\theta + \phi)$$

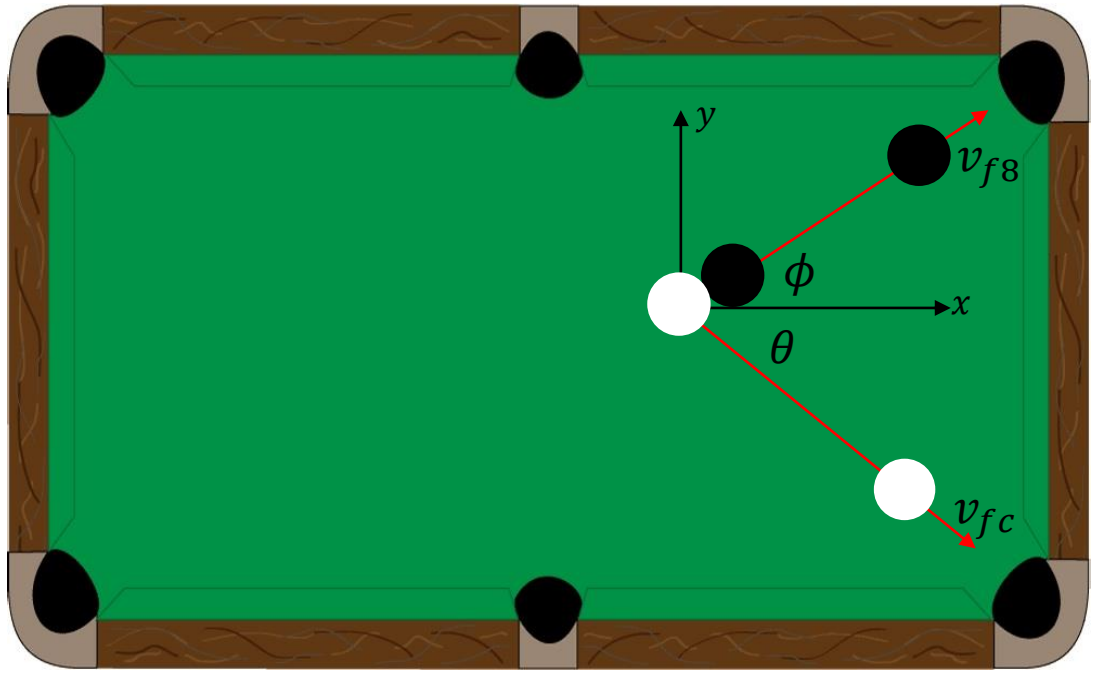
$$\theta + \phi = 90^0$$

Two equal mass objects will always scatter at 90^0. This applies *ONLY* to two equal masses. If the masses are unequal, we have to go back to conservation of momentum and kinetic energy.

Momentum and Collisions

Example 8 Solution continued:

- Now we can solve the problem for the remaining unknowns.
- If we assume that $\theta = 35^0$, then $\phi = 90^0 - \theta = 90^0 - 35^0 = 55^0$.
- Using the equations for the momenta we can calculate the final speeds.
- From the y-momentum:



$$0 = -v_{fc}\sin\theta + v_{f8}\sin\phi$$

$$v_{fc} = \frac{\sin\phi}{\sin\theta}v_{f8} = \frac{\sin 55}{\sin 35}v_{f8} = 1.42v_{f8}$$

From the x-momentum:

$$v_{ic} = v_{fc}\cos\theta + v_{f8}\cos\phi = 1.42v_{f8}\cos\theta + v_{f8}\cos\phi$$

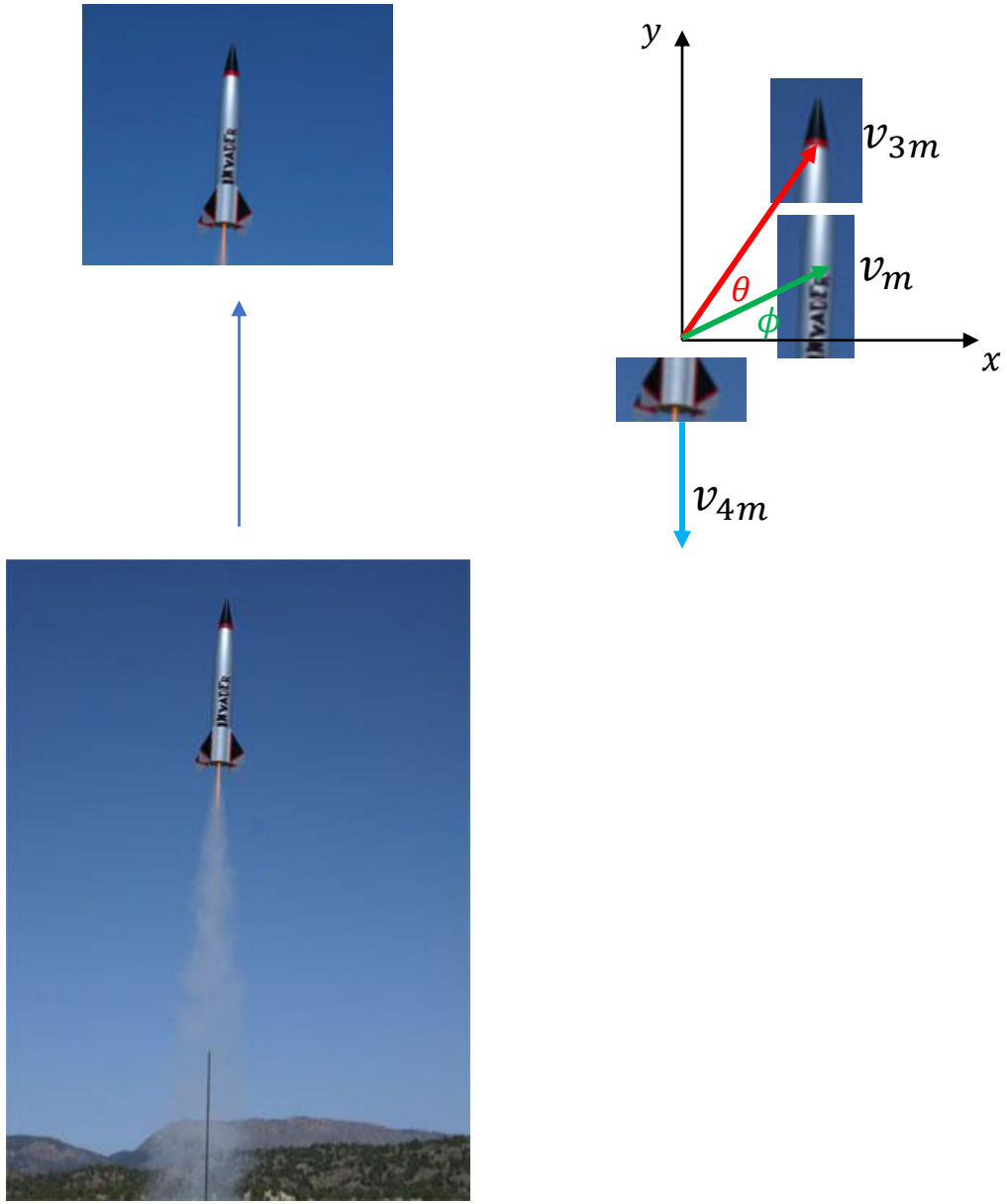
$$\longrightarrow \quad v_{f8} = \frac{v_{ic}}{1.42\cos\theta + \cos\phi} = \frac{3\frac{m}{s}}{1.42\cos 35 + \cos 55} = 1.7\frac{m}{s} \quad \longrightarrow \quad v_{fc} = 1.42v_{f8} = 1.42\times 1.7\frac{m}{s} = 2.5\frac{m}{s}$$

Momentum and Collisions

Example 9:

A model rocket used for fireworks explodes just when it reaches its highest point in a vertical trajectory. It initially bursts into three fragments with masses of m, $3m$, and $4m$, each of these to explode slightly later.

If the $4m$ fragment falls vertically downward with an initial velocity of $8\frac{m}{s}$, and the $3m$ fragment is ejected with a velocity of $10\frac{m}{s}$ at an angle of 30^0 above the horizontal, what is the velocity of the third fragment of mass m?



https://www.toysperiod.com/blog/scale-models/model-rockets-from-history-of-rocketry-to-building-your-own/

Momentum and Collisions

Example 9 Solution:

- We start with the model rocket at its highest location, which means $v_{fx} = v_{fy} = 0$, and apply conservation from the point just before the model rocket explodes to the point just after the model rocket explodes.

- We'll assume that the piece with mass m has a momentum in the same quadrant as the piece with mass $3m$. Let the magnitude of the momentum be mv and be oriented at an angle ϕ measured counter-clockwise with respect to the positive x-axis.

- This may not be where it's momentum actually is, but the mathematics will let us figure out where it is.
- Conservation of momentum in the horizontal direction:

  $\Delta p_x = p_{fx} - p_{ix} = 0$

  $p_{ix} = p_{fx} \rightarrow 0 = 3mv_{f3mx} + mv_{fmx} \rightarrow 3m\left(10\frac{m}{s}\cos 30\right) + mv_m \cos\phi \rightarrow v_m \cos\phi = -25.98\frac{m}{s}$
- Conservation of momentum in the vertical direction:

  $\Delta p_y = p_{fy} - p_{iy} = 0$

  $p_{iy} = p_{fy} \rightarrow 0 = -4mv_{4m} + 3mv_{f3my} + mv_{fmy} \rightarrow -4m\left(8\frac{m}{s}\right) + 3m\left(10\frac{m}{s}\sin 30\right) + mv_m \sin\phi \rightarrow v_m \sin\phi = 17\frac{m}{s}$

Momentum and Collisions

Example 9 Solution continued:

- Here we have that the x-component of the velocity is negative, while the y-component is positive, so the momentum vector lies in the 2nd quadrant.
- Our guess was incorrect, but that's ok. Now we know.
- Taking the ratio of these two equations gives us the angle ϕ.

$$\frac{v_m \sin\phi}{v_m \cos\phi} = \tan\phi = \frac{17\frac{m}{s}}{-25.98\frac{m}{s}} = 0.6543 \rightarrow \phi = 33.2^0$$

- To determine the magnitude of the velocity v_m for the mass m we use either equation from conservation of momentum.

$$v_m \cos\phi = \left|-25.98\frac{m}{s}\right| \rightarrow v_m = \frac{25.98\frac{m}{s}}{\cos 33.2} = 31.1\frac{m}{s}$$

$$v_m \sin\phi = 17\frac{m}{s} \rightarrow v_m = \frac{17\frac{m}{s}}{\sin 33.2} = 31.1\frac{m}{s}$$

We could also have obtained the same result using the Pythagorean theorem since we know the horizontal and vertical components of the velocity vector.

$$v_m = \sqrt{(v_m \cos\phi)^2 + (v_m \sin\phi)^2} = \sqrt{\left(-26\frac{m}{s}\right)^2 + \left(17\frac{m}{s}\right)^2} = 31.1\frac{m}{s}$$

Momentum and Collisions

Example 10:

- A karate expert strikes downward with her fist of mass $m_{fist} = 0.7kg$ breaking a stationary brick with $m_{brick} = 3.2kg$. The stiffness constant for the brick is $k = 2.6\times10^{6}\frac{N}{m}$ and the brick breaks at a deflection $d = 1.5mm$.

  a. How much work did the brick do in bringing your hand to rest, just as the brick breaks?

  b. How fast are *the brick and your fist* ($v_{fist+brick}$) moving just after you strike the brick.

  c. With what minimum speed would the karate expert's hand be moving (v_{fist}) before it collides with the brick so that she can break this karate brick? (Hint: just after your strike the brick, your hand and brick are moving at the same speed.)

Momentum and Collisions

Example 10 Solutions:

a. How much work did the brick do in bringing your hand to rest, just as the brick breaks?

$$W = -\Delta U_s = -\left(\frac{1}{2}ky_f^2 - \frac{1}{2}ky_i^2\right) = -\frac{1}{2}ky_f^2 = -\frac{1}{2}\cdot 2.6\times10^6\frac{N}{m}\cdot(1.5\times10^{-3}m)^2 = 2.93J$$

Here I'm going to ignore the small change in gravitational potential energy.

$$\Delta U_g = mgy = 0.7kg\cdot 9.8\frac{m}{s^2}\cdot 1.5\times10^{-3}m = 0.01J$$

b. How fast are *the brick and your fist* ($v_{fist+brick}$) moving just after you strike the brick.

We apply conservation of energy after the collision to determine the speed of her fist and brick after the collision, again ignoring the small change in gravitational potential energy.

$$\Delta E = 0 = \Delta K + \Delta U_g + \Delta U_s = \left(\frac{1}{2}m_{hand+brick}\overset{0}{\cancel{v_{f,hand+brick}^2}} - \frac{1}{2}m_{hand+brick}v_{i,hand+brick}^2\right) + \left(\frac{1}{2}ky_f^2 - \overset{0}{\cancel{\frac{1}{2}ky_i^2}}\right)$$

$$v_{i,hand+brick} = \sqrt{\frac{k}{m_{hand+brick}}}y_f = \sqrt{\frac{2.6\times10^6\frac{N}{m}}{0.7kg+3.2kg}}\cdot 1.5\times10^{-3}m = 1.23\frac{m}{s}$$

Momentum and Collisions

Example 10 Solutions:

c. With what minimum speed would the karate expert's hand be moving (v_{fist}) before it collides with the brick so that she can break this karate brick? (Hint: just after your strike the brick, your hand and brick are moving at the same speed.)

We apply conservation of momentum in the collision:

$$p_{iy} = p_{fy} \rightarrow m_{hand}v_{i,hand} = (m_{hand} + m_{brick})v_{f,hand+brick} \rightarrow v_{i,hand} = \left(\frac{m_{hand} + m_{brick}}{m_{hand}}\right) v_{f,hand+brick}$$

$$v_{i,hand} = \left(\frac{0.7kg + 3.2kg}{0.7kg}\right) \cdot 1.23\frac{m}{s} = 6.8\frac{m}{s}$$