Physics 111 Fall 2007 Electric Currents and DC Circuits

1. What is the average current when all the sodium channels on a 100 μ m² patch of muscle membrane open together for 1 ms? Assume a density of 50 sodium channels per μ m² of surface and a flow rate of 1000 ions per ms through each channel.

The average current is given by

$$I = \frac{\Delta Q}{\Delta t} = \frac{\frac{50 \text{ channels}}{1 \mu m^2} \times \frac{1000 \text{ ions}}{\text{ channel}} \times \frac{1.6 \times 10^{-19} \text{ C}}{\text{ ion}} \times 100 \mu m^2}{1 \times 10^{-3} \text{ s}} = 8 \times 10^{-10} \text{ A} = 0.8 \text{ nA}$$

2. The immediate cause of many deaths is ventricular fibrillation, an uncoordinated quivering of the heart as opposed to proper beating. An electric shock to the chest can

cause momentary paralysis of the heart muscle, after which the heart will sometimes start organized beating again. A *defibrillator* is a device that applies a strong electric shock to the chest over a time interval of a few milliseconds. The device contains a capacitor of several microfarads, charged to several



thousand volts. Electrodes called paddles, about 8 cm across and coated with conducting paste, are held against the chest on both sides of the heart. Their handles are insulated to prevent injury to the operator, who calls "Clear!" and pushes a button on one paddle to discharge the capacitor through the patient's chest. Assume that an energy of 300 J is to be delivered from a $30.0-\mu$ F capacitor.

a. To what potential difference must it be charged?

Consider the following, a *defibrillator* connected to a 32 μ F capacitor and a 47 k Ω resistor in a RC circuit. The circuitry in this system applies 5000 V to the RC circuit to charge it.

- b. What is the time constant of this circuit?
- c. What is the maximum charge on the capacitor?

- d. What is the maximum current in the circuit during the charging process?
- e. What are the charge and current as functions of time?
- f. How much energy is stored in the capacitor when it is fully charged?

s
$$\Delta V = \sqrt{\frac{2U}{C}} = \sqrt{\frac{2(300 \text{ J})}{30 \times 10^{-6} \text{ C/V}}} = \boxed{4.47 \times 10^3 \text{ V}}$$

- b. The time constant is given as $\tau = RC = 47 \times 10^3 \Omega \times 32 \times 10^{-6} F = 1.5 s$.
- c. The maximum charge on the capacitor is the product of the capacitance and the maximum voltage, which is

$$Q_{\rm max} = CV_{\rm max} = 32 \times 10^{-6} F \times 5000V = 0.160C$$

d. The maximum current is given from Ohm's Law

$$I_{\max} = \frac{V_{\max}}{R} = \frac{5000V}{47 \times 10^3 \Omega} = 0.106A$$

e. The charge and currents are thus given

as
$$Q(t) = 0.160C \left(1 - e^{-\frac{t}{1.5s}} \right)$$
 and $I(t) = 0.106A \left(1 - e^{-\frac{t}{1.5s}} \right)$.

f. The maximum amount of energy stored when the capacitor is fully charged

is
$$E_{\text{max}} = \frac{1}{2}CV_{\text{max}}^2 = \frac{Q_{\text{max}}^2}{2C} = \frac{1}{2}Q_{\text{max}}V_{\text{max}} = 400J$$

- 3. An immersible heater coil is to be designed to heat an insulated container with 4 liters of distilled water from 20° to 50° C in less than 30 minutes.
 - a. How much energy must be input to heat the water to this temperature?
 - b. To heat the water, what minimum power must be supplied?
 - c. If a 12 V power supply is to be used, what minimum current must flow in the heating coil?
 - d. What must be the total resistance of the heating coil? Is this a maximum or minimum resistance to heat the water in 30 minutes or less?

First we need to convert liters into kilograms. The conversion is that one liter of water has a mass of l kg. Therefore we have 4kg of distilled water that we need to heat.

a. The amount of heat energy needed to heat the water is

$$Q = m_{water} c_{water} \Delta T = 4kg \times \frac{1000cal}{1kg} \times \frac{4.186J}{1cal} \times 30^{\circ} C = 5.02 \times 10^{5} J$$

b. The minimum power that must be supplied is

$$P = \frac{\text{Energy Input}}{\text{time}} = \frac{5.02 \times 10^{8} J}{1800 s} = 278.9 W$$

- c. The minimum current that must flow is found from $P = IV \rightarrow I = \frac{278.9W}{12V} = 23.2A$.
- d. The resistance needed is $P = I^2 R \rightarrow R = \frac{278.9W}{(23.2A)^2} = 0.52\Omega$. For the minimum current this represents a maximum resistance.
- 4. Analyze the circuit shown below to find the currents flowing through and the power generated in each resistor.



- For the 1.5 k Ω and 2.5 k Ω resistors, we can add them since they are in series to get the equivalent resistance of that branch to be 4 k Ω .
- Next, the two 4 k Ω resistors are in parallel and the equivalent resistance for these two are given as the reciprocal of the sum of the reciprocals of the individual resistances. This gives an equivalent resistance of 2 k Ω .
- Lastly this leaves a battery and two resistors in series, namely 1.5 k Ω and 2 k Ω . These last two resistors are in series and the equivalent resistance is 3.5 k Ω .

Therefore we have a 3.5 k Ω equivalent circuit resistance and by Ohm's Law the current developed by the battery is $I_{circuit} = \frac{V}{R_{eq}} = \frac{6V}{3500\Omega} = 1.7 \times 10^{-3} A = 1.7 mA$.

Now working backwards through the circuit, this total current has to flow through the 1.5 $k\Omega$ resistor and the power dissipated across this resistor

is
$$P_{1.5k\Omega} = I_{1.5k\Omega}R = (1.7mA)^2 \times 1.5k\Omega = 4.4mW$$
.

- This 1.7 mA current has to flow through the remainder of the circuit which has an equivalent resistance of 2 kΩ. The potential drop across this resistor is given from Ohm's law $V_{2k\Omega} = I_{2k\Omega}R = 1.7mA \times 2k\Omega = 3.4V$. However, this branch of the circuit, expanding the circuit back to its original state, has a 4 kΩ resistor. Thus, the current through this resistor is given by using Ohm's Law again, knowing the potential drop across the resistor. Thus we find $V_{4k\Omega} = I_{4k\Omega}R \rightarrow I_{4k\Omega} = \frac{3.4V}{4k\Omega} = 0.85mA$ and the power dissipated across the 4 kΩ resistor is $P_{4k\Omega} = I_{4k\Omega}R = (0.85mA)^2 \times 4k\Omega = 2.89mW$.
- Finally, we know that 1.7 mA of current was developed by the circuit and that 0.85 mA flowed out of the junction through the 4 k Ω resistor. Thus, 0.85 mA must flow to the bottom branch to conserve charge. This gives the current through the 1.5 k Ω and 2.5 k Ω resistors as 0.85 mA and the power dissipated across each

as
$$P_{1.5k\Omega} = I_{1.5k\Omega}R = (0.85mA)^2 \times 1.5k\Omega = 1.1mW$$
 and
 $P_{2.5k\Omega} = I_{2.5k\Omega}R = (0.85mA)^2 \times 2.5k\Omega = 1.8mW$ respectively

As a last check, the power developed by the circuit is given

as $P_{circuit} = I_{circuit}R_{eq} = (1.7mA)^2 \times 3.5k\Omega = 10.12mW$. Since energy has to be conserved in the circuit we can add all of the energy (per unit time) that was dissipated across the circuit and we find 10.19 mW, with the discrepancy due to rounding errors.

5. *RC* time constants can be easily estimated by measuring the time (known as the halftime) for the capacitor voltage to decrease to half of some arbitrary starting value when discharging through a resistor. Further, we know that the voltage across the capacitor will vary as $V(t) = V_o e^{-\frac{t}{RC}}$. Show how a single measurement of the halftime can be used to determine the *RC* time constant. (Hint: substitute $V(t) = V_o/2$)

$$V(t) = V_o e^{-\frac{t}{RC}} \rightarrow \frac{V_o}{2} = V_o e^{-\frac{t}{RC}} \rightarrow \ln(\frac{1}{2}) = -\frac{t}{RC} \rightarrow t = RC \ln 2$$
. Thus measuring the time it

takes for the capacitor to discharge to have of it's initial potential, we see that we need only one measurement to calcualte the time constant.

- 6. A simple *RC* series circuit has a 100 μ F capacitor.
 - a. If the time constant is 50 s, what is the value of the resistor?
 - b. Suppose that a second identical resistor is inserted in series with the first. What is the new time constant of the circuit?
 - c. Suppose the second identical resistor is placed in parallel with the first resistor, still connected to the capacitor. What is the new time constant in this case?
 - d. Suppose that we use the single resistor from part a, but now a second identical capacitor is connected in series with the first. What is the new time constant of the circuit?
 - e. Suppose that we use the single resistor from part a, but now a second identical capacitor is connected in parallel with the first. What is the new time constant of the circuit?
 - a. The resistance is given from $\tau = RC \rightarrow R = \frac{\tau}{C} = \frac{50s}{100 \times 10^{-6} F} = 5 \times 10^5 \Omega$.
 - b. The equivalent resistance for these two resistors in series is the sum of the individual resistances, or $1 \times 10^6 \Omega$ and the new time constant is

$$\tau_{series} = R_{series} C = 1 \times 10^6 \Omega \times 100 \times 10^{-6} F = 100 s$$
.

c. The equivalent resistance for these two resistors in parallel is the reciprocal of the sum of the reciprocals of the individual resistances, or $2.5 \times 10^5 \Omega$ and the new time constant

is
$$\tau_{parallel} = R_{parallel}C = 2.5 \times 10^5 \Omega \times 100 \times 10^{-6} F = 25 s$$

- d. The equivalent capacitance for these two capacitors in series is the reciprocal of the sum of the reciprocals of the individual capacitances, or 5.0×10^{-5} F and the new time constant is $\tau_{series} = RC_{series} = 5.0 \times 10^5 \Omega \times 5.0 \times 10^{-5} F = 25s$.
- e. The equivalent resistance for these two capacitors in parallel is the sum of the individual capacitances, or $200 \times 10^{-6} \Omega$ and the new time constant is

$$\tau_{parallel} = RC_{parallel} = 5.0 \times 10^5 \Omega \times 200 \times 10^{-6} F = 100s$$

- 7. A 100 μ F capacitor wired in a simple series *RC* circuit is initially charged to 10 μ C and then discharged through a 10 k Ω resistor.
 - a. What is the time constant of the circuit?
 - b. What is the initial current that flows?
 - c. How much charge is left on the capacitor after 1 time constant?
 - d. What is the current after 1 time constant?
 - e. How much charge is left on the capacitor after 3 time constants have elapsed and what current is flowing then?
 - a. The time constant is the product of the resistance and the capacitance, or 1.0s.
 - b. The initial current that flows is found from Ohm's Law

$$V_C = \frac{Q}{C} = IR \rightarrow I = \frac{Q}{RC} = \frac{10 \times 10^{-6}C}{1s} = 10 \times 10^{-6}A = 10 \,\mu A$$
.

c. After 1 time constant we have
$$Q(1s) = Q_o e^{-\frac{1}{1s}} = \frac{10\mu C}{c} = 0.368 \times 10 \mu C = 3.68 \mu C$$
.

d. The current that flows after one time constant is

$$I(1s) = I_o e^{-\frac{1s}{1s}} = \frac{10\,\mu A}{e} = 0.368 \times 10\,\mu A = 3.68\,\mu A$$

e. After 3 time constants we have

$$Q(3s) = Q_o e^{-\frac{3s}{1s}} = \frac{10\mu C}{e^3} = 0.050 \times 10\mu C = 0.50\mu C \text{ and the current that is flowing is}$$
$$I(3s) = I_o e^{-\frac{3s}{1s}} = \frac{10\mu A}{e^3} = 0.050 \times 10\mu A = 0.50\mu A.$$

8. Find the reading that each of the (ideal) meters would have in the following circuit.



For the 1.0 k Ω and 5 k Ω resistors, we can add them since they are in series to get the equivalent resistance of that branch to be 6 k Ω .

- In the middle branch we have a 3 k Ω resistor in series with a 10 k Ω resistor. The equivalent resistance of this branch is the sum of the individual resistors, or 13 k Ω .
- Next, we have the lower two branches in parallel. Thus the 6 k Ω resistor is in parallel with the 13 k Ω resistor and the equivalent resistance for these two are given as the reciprocal of the sum of the reciprocals of the individual resistances. This gives an equivalent resistance of 4.11 k Ω .
- Lastly this leaves a battery and two resistors in series, namely 4.11 k Ω and 1 k Ω . These last two resistors are in series and the equivalent resistance is5.11 k Ω . Therefore we have a 5.11 k Ω equivalent circuit resistance and by Ohm's Law the current developed by the battery and passes through ammeter <u>A₂</u> is

$$I_{circuit} = \frac{V}{R_{eq}} = \frac{10V}{5110\Omega} = 1.96 \times 10^{-3} A \approx 2.0 mA.$$

- This current passes through the 1 kW resistor, so the potential drop across it as displayed by voltmeter V_l is given by Ohm's Law $V_{lk\Omega} = IR = 2.0mA \times 1k\Omega = 2V$.
- The potential drop across the bottom two branches must be the same and is 8V (10V 2V) by Kirchoff's rule for potential drops.)

The current that flows through the middle branch is given by Ohm's Law with an equivalent resistance of 13 k Ω and that current is $I = \frac{V}{R} = \frac{8V}{13k\Omega} = 0.62mA$.

By the junction rule, 2mA of current flows into the junction of the middle and bottom branch, and 0.62 mA flows through the middle branch so a current of 1.38mA must flow through the bottom branch as well as through ammeter A_1 .

Lastly we need the potential drop across the 10 k Ω resistor in the middle branch. The current in this branch is 0.62 mA and by Ohm's Law $V_{10k\Omega} = IR = 0.62mA \times 10k\Omega = 6.2V$, which is what would be displayed on voltmeter V_2 .

- Two 100 m 14 gauge wires (1.63 mm diameters), one of copper and one of aluminum, are soldered together and the 200 m wire is then connected to a 6 V dc power supply with unlimited current.
 - a. How much current flows in the wire?
 - b. What is the potential across each 100 m section of wire?
 - c. How much power is developed in each section of wire?
 - d. How much total power was delivered by the battery? Does this equal the power dissipated by the circuit? If not, where did the extra energy go?

a. To determine the current that flows in the wire we need the equivalent resistance for theses two sections of wire. Let's model them as two resistors in series. The equivalent resistance is the sum of the individual resistances that we now need to calculate. For the

copper section we have
$$R_{Cu} = \rho_{Cu} \frac{L_{Cu}}{A_{Cu}} = 1.78 \times 10^{-8} \Omega m \times \frac{100m}{\pi (0.815 \times 10^{-3} m)^2} = 0.815 \Omega$$

while for the aluminum

section
$$R_{Al} = \rho_{Al} \frac{L_{Al}}{A_{Al}} = 2.8 \times 10^{-8} \Omega m \times \frac{100m}{\pi (0.815 \times 10^{-3} m)^2} = 1.34 \Omega$$
. Thus the

equivalent resistance is $R_{eq} = R_{Cu} + R_{Al} = 2.16\Omega$. Thus the current that flows through both sections $I = \frac{V}{R_{eq}} = \frac{6V}{2.16\Omega} = 2.78A$.

- b. The potential difference across each section is given for copper, $V_{Cu} = IR_{Cu} = 2.78A \times 0.815\Omega = 2.27V$ and $V_{Al} = IR_{Al} = 2.78A \times 1.34\Omega = 3.73V$ for aluminum. Note, that the sum of the potential drops across each resistor sums to 6V as it should.
- c. The power dissipated in the copper section is $P_{Cu} = I^2 R_{Cu} = (2.78A)^2 \times 0.815\Omega = 6.3W$, while for aluminum $P_{Al} = I^2 R_{Al} = (2.78A)^2 \times 1.34\Omega = 10.4W$.
- d. The power developed by the battery is $P = I^2 R = (2.78A)^2 \times 2.16\Omega = 16.7W$, which is equivalent to the power dissipated across each section of resistor.
- 10. The Earth's atmosphere is able to act as a capacitor, with one plate the ground and the other the clouds and in between the plates an air gap. Air, however, is not a perfect insulator and can be made to conduct, so that the separation of charges from the cloud to ground can be bridged. Such an event is called a lightning strike. For this question, we'll model the atmosphere as a spherical capacitor and try to calculate the number of lightning strikes that happen every day. First, let's assume that the clouds are distributed around the entire Earth at a distance of 5000 m above the ground of area $4pR^2_{Earth}$, where $R_{Earth} = 6400$ km.
 - a. What is the resistance of the air gap?

Next we need to calculate the capacitance of the Earth-cloud capacitor. Here we will use the fact that the capacitance $C = \frac{Q}{\Delta V}$ and we'll calculate ΔV . Assuming that we

have a spherical charge distribution $V = k \frac{Q}{r}$, so that ΔV is the difference in potential

between the lower plate (the Earth's surface) and the upper plate (the clouds) and that in a typical day, $5x10^5$ C of charge is spread over the surface of the Earth.

b. What is ΔV ?

c. What is the capacitance of the Earth-cloud capacitor?

Since the accumulated charge will dissipate through the air, we have a simple RC circuit.

d. What is the time constant for this discharge that is spread over the whole surface of the earth?

Experimentally it is found that for each lightning strike about 25 C of negative charge is delivered to the ground.

- e. What is the number of lightning for this amount of charge?
- f. Approximately how long would it take the Earth-cloud capacitor to discharge to 0.3% of its initial amount?

So to answer the question of how many lightning strikes per day, we know that we get part e number of strokes in part f amount of time.

g. How many lightning strokes per day?

a. The resistance of the air-gap is given

as
$$R = \rho \frac{L}{A} = (3 \times 10^{13} \Omega m) \frac{5 \times 10^3 m}{4\pi (6.4 \times 10^6 m)^2} = 291 \Omega$$
.

b. Assuming that the Earth-cloud system can be modeled as two spherical charge distributions separated by a fixed distance, we find the difference in potential to be

$$\Delta V = V_{lower} - V_{upper} = k \left(\frac{Q}{r_{lower}} - \frac{Q}{r_{upper}} \right) = 9 \times 10^9 \frac{Nm^2}{C^2} \times 5 \times 10^5 C \times \left(\frac{1}{r_{earth}} - \frac{1}{r_{earth}} + 5000m \right) = 4.88 \times 10^5 V$$

c. The capacitance is therefore $C = \frac{Q}{\Delta V} = \frac{5 \times 10^5 C}{4.88 \times 10^5 V} = 1.03 F$.

d. The time constant is the product of the resistance and capacitance of the circuit. Thus the time constant is $\tau = RC = 291\Omega \times 1.03F = 298.2s$.

e. For each lightning strike, 25C of charge is delivered. Thus the number of lightning strikes is the total charge divided by 25 C per strike, or 20,000 strikes.

To calculate the time, we use the equation for a discharging capacitor. f.

$$Q(t) = Q_{\max} e^{-\frac{t}{RC}} \to 0.003 = e^{-\frac{t}{298.2s}} \to t = 1732.3s = 0.48hr$$
.

g. So we get 20,000 strikes every 0.48hrs, or 41,667 strikes per hour. Thus in 24 hours we get about 1,000,000 strikes. This value, even if only approximate, is not far from the observed number of about 1 million per day!!

11. The student engineer of a campus radio station wishes to verify the effectiveness of the lightning rod on the antenna mast as shown below. The unknown resistance R_x is between points C and E. Point E is a true ground but is inaccessible for direct measurement because this stratum is several meters below the



Earth's surface. Two identical rods are driven into the ground at A and B, introducing an unknown resistance R_{y} . The procedure is as follows. Measure resistance R_{1} between points A and B, then connect A and B with a heavy conducting wire and measure resistance R_2 between points A and C.

- (a) Derive an equation for R_x in terms of the observable resistances, R_1 and R_2 .
- (b) A satisfactory ground resistance would be $R_x < 2.00 \ \Omega$. Is the grounding of the station adequate if measurements give R_1 = 13.0 **Ω** and $R_2 = 6.00$ **Ω**?



(a)

For the first measurement, the equivalent circuit is as shown in Figure 1.

$$R_{ab} = R_1 = R_y + R_y = 2R_y$$

 $R_y = \frac{1}{2}R_1.$

so



For the second measurement, the equivalent circuit is shown in Figure 2.

Thus,
$$R_{ac} = R_2 = \frac{1}{2}R_y + R_x$$
. (2)

Substitute (1) into (2) to obtain:

$$R_2 = \frac{1}{2} \left(\frac{1}{2} R_1 \right) + R_x$$
, or $R_x = R_2 - \frac{1}{4} R_1$.

(b) If $R_1 = 13.0 \Omega$ and $R_2 = 6.00 \Omega$, then $R_x = 2.75 \Omega$

The antenna is inadequately grounded since this exceeds the limit of 2.00Ω .