

Electrostatic Forces and Fields

Coulomb's Law

Electric charge is an intrinsic property of matter, in exactly the same fashion as mass. At the atomic level electric charge comes in three types and is carried by the three elementary particles. These charges are the proton (a positive electric charge), the electron (a negative electric charge), and the neutron (a zero electric charge.) These elementary particles are the building blocks for all matter and typical atomic structure as revealed by experiments has a heavy dense nucleus (composed of protons and neutrons) surrounded by electrons in specific probability distributions, or electron clouds called allowed orbitals. Atoms in general are electrically neutral meaning that the number of protons is equal to the number of electrons, but since the electrons are loosely bound the nucleus, they may be easily added to or removed from the atom. Ionization of an atom is the process by which electrons are added or removed with minimal inputs of energy. If electrons are removed from an electrically neutral atom, a negative ion is formed. Conversely, if electrons are added to an electrically neutral atom, a positive ion is formed.

Experiments with electric charge demonstrate a number of phenomena. The first of which is that the magnitude of the charge on the proton and the electron are the same and this value, called the smallest amount of free charge found in nature, is found to be that of the charge on the electron. Thus one elementary charge has a magnitude of $|p| = |e^-| = e = 1.6 \times 10^{-19} \text{ C}$. There are several implications that can be made here, and can be seen to hold true experimentally. The first is that electric charge is quantized, or

occurs in only discrete quantities, namely $Q_{total} = n|e^-|$, where n is the number of discrete elementary charges, e , that make up the total charge Q_{total} . The second implication that can be made is there could be a collection of more elementary charges that exist only in bound states and these could make up an elementary charge. For example, it is found that quarks are fundamental particles that are the constituent particles out of which that make up protons and neutrons are made. Quarks exhibit fractional electric charges in units of $\pm \frac{1}{3}e$ or $\pm \frac{2}{3}e$ and never occur as a single free unit of charge. Quarks only occur as part of a bound system.

In addition, it is experimentally found that if two charges Q_1 and Q_2 , carry the same algebraic sign, the charges will be repelled away from each other, while if they carry the opposite algebraic sign they will be attracted. This implies that there must be a force that exists between electric charges. This form of this force law was experimentally determined by Coulomb in the 1770's and bears his name. Coulomb's experiment involved placing amounts of charge, Q_1 and Q_2 on two isolated insulating spheres separated by a known distance r . A charge Q_1 was placed on a dumbbell that was free to rotate about a wire through the center of the dumbbell. The second insulating sphere with charge Q_2 was brought close to Q_1 and would cause Q_1 to rotate about the wire. The torque that was created is related to the magnitude of the electrostatic force that exists between Q_1 and Q_2 . Coulomb found two important results from his experiments. The first, given a fixed amount of charge Q_1 and Q_2 on each of the spheres, the magnitude of the electrostatic force varied as the inverse of the square of the distance between the centers of the two charges. Second, for a fixed distance between the charges, the magnitude of the electrostatic force varied as the product of the two charges on the

spheres. Coulomb's results can be summarized as follows: $F_{\text{electrostatic}} \propto \frac{Q_1 Q_2}{r_{1,2}^2}$, where $r_{1,2}$

is the center to center distance between Q_1 and Q_2 . To make this result an equality rather than a proportionality, a constant is multiplied on the right hand side, and this constant of proportionality is called Coulomb's constant and has a value of $8.99 \times 10^9 \frac{\text{Nm}^2}{\text{C}^2}$. This value can be experimentally verified by recreating Coulomb's experiment as is sometimes done in undergraduate physics laboratory classes, keeping either the charges fixed (and varying the distance between the charges) or keeping the distance fixed (and varying the amount of charge placed on each sphere.) In either case, graphical analysis will then yield a value for Coulomb's constant. Therefore, Coulomb's Law for static distributions of

charge is generally written as $\vec{F}_{\text{electrostatic}} = k \frac{Q_1 Q_2}{r_{1,2}^2} \hat{r}$, where, \hat{r} is a unit vector that points

along the line joining the two charges. In other words, Coulomb's Law gives the magnitude of the force that exists between the two charges and always points along the line joining the two charges. If there is more than one charge present, Coulomb's Law still applies and the electrostatic force on one charge due to all other charges is the vector sum of all of the forces on that charge due to the other charges present. As one final comment, Coulomb's Law is applicable to point charges (objects that are much smaller than the distance between them) that are at rest (static.)

Example #1 – How big is a Coulomb of Charge?

Suppose that a two insulating balls are separated by 1.0 m and that a charge of 1.0 C is placed on each. What is the magnitude of the electrostatic force felt by one of the spheres?

Solution – Applying Coulomb’s Law we find

$$F = k \frac{Q_1 Q_2}{r_{1,2}^2} = 8.99 \times 10^9 \frac{Nm^2}{C^2} \times \frac{1.0C \times 1.0C}{(1.0m)^2} = 8.99 \times 10^9 N .$$

This equates to a weight of about 2.2×10^9 pounds, given that a weight of 1 N $\sim \frac{1}{4}$ pound. Here we can draw the conclusion that one Coulomb of charge is a huge amount and in general we will only see fractions of Coulombs of charge in practice. In other words we will in general see micro-Coulombs ($1\mu C = 1 \times 10^{-6} C$) or nano-Coulombs ($1nC = 1 \times 10^{-9} C$) of charge.

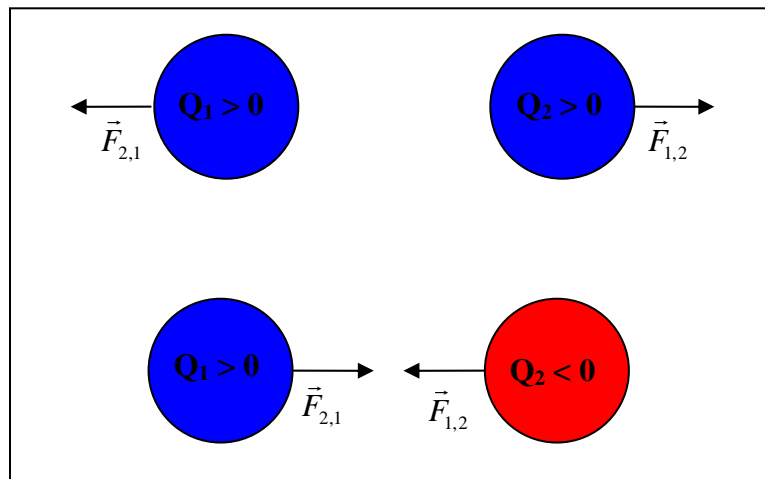
Example #2 – How many elementary charges are in 1 μC of charge?

Solution – We take 1 μC of charge and divide this by one elementary charge.

$$\#e^- = 1\mu C \times \frac{1e^-}{1.6 \times 10^{-19} C} = 6.25 \times 10^{12} e^-$$

Coulomb’s Law for the electrostatic force is also a Newton’s Third Law force.

To see this, we consider the arrangement of charges shown below and explicitly write out the forces that act on each charge.



From this diagram we see that for any two charges, the forces exerted on the charges are equal in magnitude and oppositely directed and thus we have $\vec{F}_{1,2} = -\vec{F}_{2,1}$, which is Newton's Third Law.

Early on in the discussion of electric charge we said that there are three types of charge. Let's suppose that we have a positive electric charge (a proton) that is separated from a negative electric charge (an electron) as would be typically found in a hydrogen atom, and let us calculate the electrostatic force felt by either the proton or the electron.

Example #3 – What is the magnitude of the electrostatic force between a proton and an electron in a hydrogen atom?

Solution – To calculate the magnitude of the electrostatic force that exists between a proton and an electron in a hydrogen atom, we need to know the separation of the two charges. Typically for a hydrogen atom in its ground state, we have the separation between the proton and the electron, known as one Bohr radius, $0.53 \times 10^{-10} \text{ m}$. Applying Coulomb's Law we find for the magnitude of the electrostatic force

$$F_{\text{electrostatic}} = k \frac{Q_1 Q_2}{r_{1,2}^2} = k \frac{e \times e}{r_{p,e}^2} = 8.99 \times 10^9 \frac{\text{Nm}^2}{\text{C}^2} \times \frac{(1.6 \times 10^{-19} \text{ C})^2}{(0.53 \times 10^{-10} \text{ m})^2} = 8.2 \times 10^{-8} \text{ N} .$$

The direction of the electrostatic force that the electron feels due to the proton is directed towards the proton and has a magnitude of $8.2 \times 10^{-8} \text{ N}$, while the electrostatic force that the proton feels due to the electron is directed toward the electron in accordance with Newton's Third Law, and has a magnitude also of $8.2 \times 10^{-8} \text{ N}$.

In general when we do problems involving forces we usually have to worry about the gravitational force exerted on the objects. So let's ask when dealing with elementary particles do I have to worry about gravitational forces? In other words, do I have to worry about the weight of these particles? To answer this question, we will calculate the magnitude of the gravitational force between the proton and the electron in a ground-state hydrogen atom and compare this to the electrostatic force calculated above.

Example #4 – The gravitational force between the proton and electron in a hydrogen atom

Solution - To calculate the magnitude of the gravitational force we need to use

Newton's Universal Law of Gravity $F_{gravity} = G \frac{M_1 M_2}{r_{1,2}^2}$. Looking up the masses of

the proton and the electron we find $M_p = 1.67 \times 10^{-27} \text{ kg}$ and $M_e = 9.11 \times 10^{-31} \text{ kg}$

respectively. Using the separation between the proton and the electron in a

hydrogen atom, we find the magnitude of the gravitational force to be

$$F_{gravity} = 6.67 \times 10^{-11} \frac{\text{Nm}^2}{\text{kg}^2} \frac{1.67 \times 10^{-27} \text{ kg} \times 9.11 \times 10^{-31} \text{ kg}}{(0.53 \times 10^{-10} \text{ m})^2} = 3.6 \times 10^{-47} \text{ N}.$$

Comparing this to the electrostatic force given above we see

that $\frac{F_{electrostatic}}{F_{gravity}} = 2.3 \times 10^{39}$, which implies that the electrostatic force is much larger

than the gravitational force, by a factor of 10^{39} . Thus we don't need to worry

about the force of gravity when working with elementary particles. This is not

true when we have other more massive objects, like coffee cups, balls, and

airplanes!

Now, what would happen if we were to somehow release the proton and the electron and allow them to move under the influence of the electrostatic force, what would happen?

Example #5 – Colliding a proton and electron

Suppose that a proton and an electron are separated by a distance d (maybe d could be the typical separation in a hydrogen atom.) Suppose further that the proton is at the origin and that the electron is to the right of the proton by this distance d . If the electron and proton are released from rest at the same time, qualitatively where will the proton and the electron collide?

Solution – The proton and the electron will experience the same magnitude of the electrostatic force. However, since the proton is more massive than the electron, by about a factor of 1881, and thus the accelerations of the proton and the electron will be different. So, let us first calculate the initial accelerations of the proton and the electron. From Newton's Second Law, the acceleration of the proton is

$$a_p = \frac{F}{m_p} = \frac{8.2 \times 10^{-8} \text{ N}}{1.67 \times 10^{-27} \text{ kg}} = 4.6 \times 10^{19} \frac{\text{m}}{\text{s}^2}, \text{ while the acceleration of the electron is}$$

$$a_e = \frac{F}{m_e} = \frac{8.2 \times 10^{-8} \text{ N}}{9.11 \times 10^{-31} \text{ kg}} = 9.0 \times 10^{22} \frac{\text{m}}{\text{s}^2}. \text{ Since the acceleration of the electron is greater}$$

than that of the proton, in the same interval of time, the electron will have a greater change in velocity and cover a larger distance than the proton. Therefore, the proton and the electron will collide closer to the proton at a distance somewhere between 0 and $d/2$.

Be Careful – This acceleration is not a constant of the motion since the force is not constant, but changes with distance. You would not want to use the classical equations of motion (valid for constant acceleration) to calculate the position of the collision.

Equations of motion for this non-constant acceleration could be developed but are beyond the scope of this text and unfortunately only a qualitative solution can be given at this point.

So far we have seen Coulomb's Law and some examples of how to calculate magnitudes of the electrostatic force that exists between two objects with charge. Now we will turn our attention to some more sophisticated problems and explore the vector nature of Coulomb's Law in which we apply it to situations involving more than two charges. To do this we will need a strategy. My strategy is as follows and this methodology will be used to solve all of the problems that involve vectors. First you need to pick a convenient coordinate system. It does not matter what that coordinate system is, but the choice should be well suited to the physical situation and you need to be consistent when assigning algebraic signs to the vector quantities based upon this coordinate system. Next, I will draw all of the vectors that represent the physical quantity of interest on the object of interest. Typically this means that I will pick a charge and draw all of the forces, say, that act on that charge. I will then break up those physical quantities represented by the vectors into their components (based on the choice of coordinate system) and sum the vectors algebraically to calculate the net components associated with a particular physical quantity in a particular direction. I will then report the result as a vector (using unit vector notation) or as a magnitude and a direction (measured with respect to some convenient starting point.)

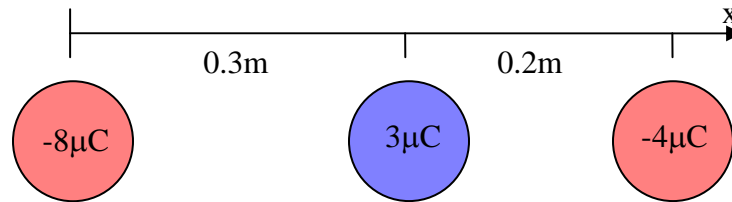
Alright, that's a whole bunch of words. How do you actually apply this method to solve problems? To answer that, we will look at two specific examples. The first is a

one-dimensional problem involving three charges in a line and the second is a two-dimensional problem involving charges located on the vertices of a right-triangle.

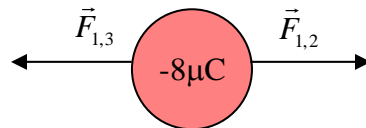
Example #6 – Three charges in a line

What is the net electrostatic force on the leftmost charge shown below due to the other two charges, where $Q_1 = -8 \mu\text{C}$, $Q_2 = 3 \mu\text{C}$, and $Q_3 = -4 \mu\text{C}$?

Solution - Given the diagram below we choose the origin of the coordinate system to be at the leftmost charge and select to the right as the positive x-direction.



Drawing the forces on the leftmost charges we can apply Coulomb's Law and write



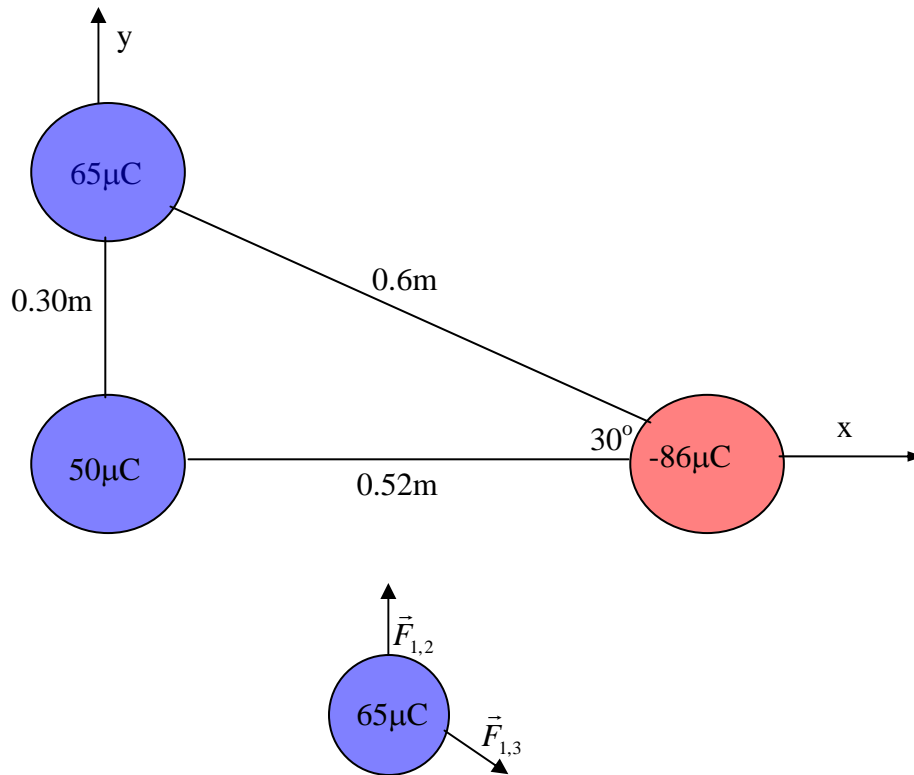
$$\vec{F}_{net,1} = \vec{F}_{1,3} + \vec{F}_{1,2} = \left(-k \frac{Q_1 Q_3}{r_{1,3}^2} + k \frac{Q_1 Q_2}{r_{1,2}^2} \right) \hat{i}$$

$$\vec{F}_{net,1} = 9 \times 10^9 \frac{\text{Nm}^2}{\text{C}^2} \times 8 \times 10^{-6} \text{C} \times \left(-\frac{4 \times 10^{-6} \text{C}}{(0.5\text{m})^2} + \frac{3 \times 10^{-6} \text{C}}{(0.3\text{m})^2} \right) = +1.2 \text{N} \hat{i}$$

The symbol $\vec{F}_{A,B}$ used here means the electrostatic force on object A due to object B. Thus the net electrostatic force on the leftmost charge is 1.2 N directed along the positive x-axis.

Example #7 – Three charges at the vertices of a right triangle

What is the net electrostatic force on the $65\ \mu\text{C}$ charge due to the other two charges shown, where $Q_1 = 65\ \mu\text{C}$, $Q_2 = 50\ \mu\text{C}$, and $Q_3 = -86\ \mu\text{C}$?



The forces on the $65\ \mu\text{C}$ charge are as shown above and we need to find the net x- and net y-forces. Thus for the net x- and y-forces we have

$$\sum F_{net,x} : k \frac{Q_3 Q_1}{r_{3,1}^2} \cos \theta = 9 \times 10^9 \frac{\text{Nm}^2}{\text{C}^2} \times \frac{86 \times 10^{-6} \text{C} \times 50 \times 10^{-6} \text{C}}{(0.6\text{m})^2} \cos 30 = 120\text{N}$$
$$\sum F_{net,y} : +k \frac{Q_2 Q_1}{r_{2,1}^2} - k \frac{Q_3 Q_1}{r_{3,1}^2} \sin \theta = 9 \times 10^9 \frac{\text{Nm}^2}{\text{C}^2} \times 65 \times 10^{-6} \text{C} \times \left[\frac{50 \times 10^{-6} \text{C}}{(0.3\text{m})^2} - \frac{86 \times 10^{-6} \text{C}}{(0.6\text{m})^2} \sin 30 \right] = 260\text{N}$$

This gives using the Pythagorean Theorem the net electrostatic force as

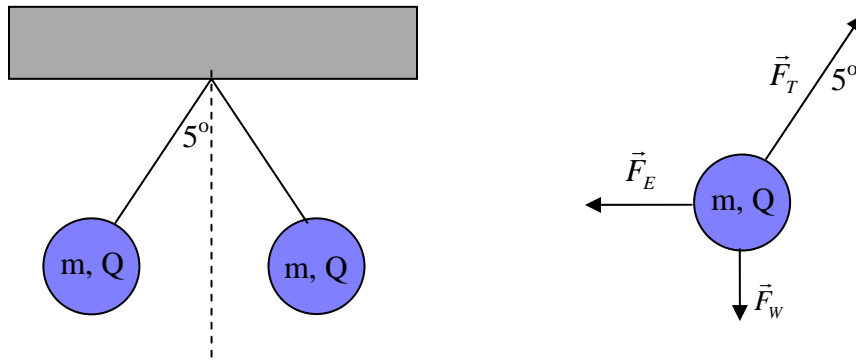
$$\vec{F}_{Net} = \sqrt{(120\text{N})^2 + (260\text{N})^2} @ \phi = \tan^{-1} \left(\frac{260\text{N}}{120\text{N}} \right) = 290\text{N} @ \phi = 65^\circ$$

Now let's apply what we've learned so far to somewhat more complicated examples.

Example #8 – How much charge does it take?

The leaves of an “*electroscope*” are constructed out of two identical balls whose mass are $3.2 \times 10^{-2} \text{ kg}$ and these balls hang in equilibrium from two strings of length $L = 15 \text{ cm}$. When in equilibrium, the balls make a 5° angle with respect to the vertical. How much charge is needed on each sphere to produce this situation?

Solution – We choose to examine the forces on the left most charge and assume a standard Cartesian coordinate system. The physical representation of the problem statement is shown below left. The forces that act on the mass are the tension force in the string, the weight of the mass, and the electrostatic force and are shown below right.



Applying Newton's 2nd Law in the horizontal and vertical directions we find

$$\sum F_x : F_T \sin 5 - F_E = ma_x = 0 \rightarrow F_E = k \frac{Q^2}{r} = F_T \sin 5$$

$$\sum F_y : F_T \cos 5 - F_W = ma_y = 0 \rightarrow F_T = \frac{mg}{\cos 5}$$

So, we have two equations and two unknowns, namely F_T and Q (from F_E .) Substituting F_T from the summation of the y-forces into the expression in the x-forces, we can solve

for Q . This produces $Q = \sqrt{\frac{mg \sin 5 r^2}{k \cos 5}} = \sqrt{\frac{mg \sin 5 (2L \sin 5)^2}{k \cos 5}} = 4.4 \times 10^{-8} C$ for the charge

needed on each sphere. In order to perform the above calculation, we have also used the fact that the charges are separated by a distance r , one-half of which can be calculated from the angle given and the length of the string. Thus we

have $\frac{r}{2} = L \sin 5 \rightarrow r = 2L \sin 5$. In addition we could also calculate the magnitude of the

tension force, $F_T = \frac{mg}{\cos 5} = \frac{3.2 \times 10^{-2} kg \times 9.8 \frac{m}{s^2}}{\cos 50} = 0.32 N$

Example #9 – The radioactive decay of ${}_{92}^{238}U$.

In the radioactive decay of uranium ${}_{92}^{238}U$, the center of the emerging alpha particle ${}_2^4He$ is, at a certain instant $9 \times 10^{-15} m$ from the center of the thorium daughter nucleus ${}_{90}^{234}Th$.

- What is the magnitude of the electrostatic force on the alpha particle? (Hint: In spectroscopic notation, ${}_Z^AX$ A represents the atomic mass in atomic mass units (amu), Z represents the atomic number, and X is the element.)

Solution - The magnitude of the electrostatic force is given by Coulomb's Law and is due to the electrostatic repulsion of the protons that make up the alpha particle and the thorium nucleus:

$$F_{\alpha,Th} = k \frac{Q_{\alpha} Q_{Th}}{r_{\alpha,Th}^2} = 9 \times 10^9 \frac{Nm^2}{C^2} \times \frac{(2e^-) \times (90e^-)}{(9 \times 10^{-15} m)^2} = 512 N$$

- What is the acceleration of the alpha particle? (Hint: $1 amu = 1.67 \times 10^{-27} kg$.)

Solution - The acceleration of the alpha particle is found from Newton's 2nd Law:

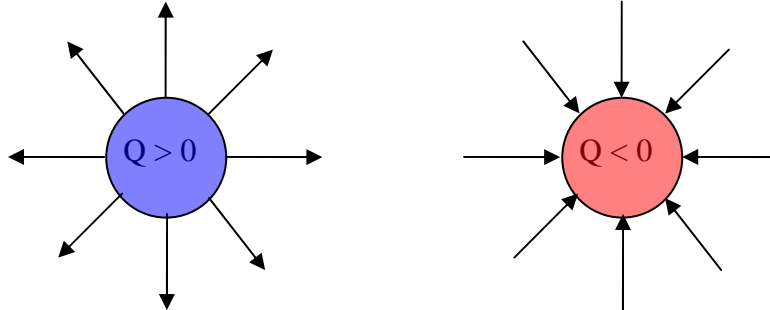
$$F_{\alpha} = m_{\alpha} a_{\alpha} = 512 N \rightarrow a_{\alpha} = \frac{512 N}{m_{\alpha}} = \frac{512 N}{4 \times 1.67 \times 10^{-27} kg} = 7.7 \times 10^{28} \frac{m}{s^2}$$

The Electric Field

Most Newton's Law forces are called contact forces because for an object to accelerate because of an applied force, the force has to be applied to the object through direct physical contact. However, some forces, like Gravity and the Electrostatic force extend over distances even when the objects do not touch. These are called "action at a distance" forces. Forces like these are exerted on masses (due to the Force of Gravity) or charges (due to the Electrostatic Force) through a field. To investigate these fields we use a test object. For gravitational fields, we use a test mass m_o while for electric fields we will use a test charge q_o . Here we will assume that the test charge is positive and very small in mass compared to the charge Q whose electric field we want to investigate. The choice of q_o being positive is a convenience and chosen to mirror the results you get from studying the motion of masses in a gravitational field. The fact that the test charge is much less massive is so that the acceleration of the test charge is much larger than the charge whose field is being investigated. Since both charges exert equal and opposite forces on each other, I would like only the test charge to move and have the field due to the other charge be static.

Since the electrostatic force will cause q_o to move, if Q is a positive charge, then q_o will feel a repulsive force from Q and move away. If, however, Q is a negative charge, then q_o will feel an attractive force and move toward the charge. The electric field lines will then point radially outward from a positive charge and point radially inward toward a negative charge as shown below.

Example #10 - The Electric field lines due to a positive and negative electric charge investigated using a small positive test charge q_0 .



Electric field lines are pictorial representations of the electric field that exists in space due to the presence of a charge. In other words they represent the lines of force felt by q_0 due to the interaction of q_0 with the electric field due to Q . Thus the electrostatic force on q_0 due to Q is exerted through q_0 's interaction with the electric field of Q . The electric field surrounds the charge and exists whether or not a test charge is present to investigate the field. This is exactly analogous to a gravitational field. A gravitational field exerts a force on a test mass m_o through m_o 's interaction with the gravitational field of a mass M and the gravitational field of M exists even if there is no test mass present to probe this field. In addition, the electric field lines indicate the direction of the electric field at any point in space due to some distribution of charge. The magnitude of the electric field, a vector quantity, has to be determined by the superposition of all the electric fields due to all charges present. We define the electric field of a point charge as the electrostatic force per unit test charge, or $\vec{E} = \frac{\vec{F}}{q_o}$, so that the electrostatic force may

be written as $\vec{F} = q_o \vec{E}$. Thus for a point charge Q , we have for the magnitude of the electric field $E = k \frac{Q}{r^2}$ and the direction is tangent to the field line at that point in space.

Electric field lines can be seen to start on positive charges, extending radially outwards, and end on negative charges, pointing radially inward. The number of electric field lines (per unit area of space) is proportional to the magnitude of the electric charge present. In other words, the flux (or number) of electric field lines through an area perpendicular to the electric field vector is proportional to the total charge enclosed by a surface of area A . This is called Gauss' Law and is used to calculate the electric field for an arbitrary distribution of charge. Gauss' Law can be written as $\oint_{\text{Surface}} \vec{E} \cdot d\vec{a} = \frac{Q_{\text{enclosed}}}{\epsilon_o}$ and is useful when studying the propagation of electromagnetic radiation, or light. We will apply Gauss' Law to a couple of situations below, only as illustrative examples, even though we will only ever use electric field of a point charge $E = k \frac{Q}{r^2}$ as given above.

Example #11 – What are the electric fields due to a point charge Q , a uniform plate of charge Q (per unit area), and a uniform line of charge Q (per unit length) using Gauss' Law?

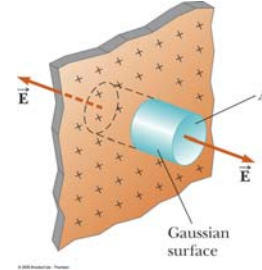
Solution

- For a point charge, Q we choose to have the vector pointing in the direction of the electric field due to a positive charge Q as shown above. First, we find that at a fixed distance away, r , the flux of field lines is constant, indicating that the electric field is a constant. Evaluating Gauss' Law we have

$$\oint_{\text{Surface}} \vec{E} \cdot d\vec{a} = E \oint_{\text{sphere of radius } r} da = 4\pi r^2 E = \frac{Q_{\text{enclosed}}}{\epsilon_o} \rightarrow E_{\text{point charge}} = \frac{Q}{4\pi\epsilon_o r^2} = k \frac{Q}{r^2}, \text{ which varies}$$

as the inverse of the distance squared from the point charge.

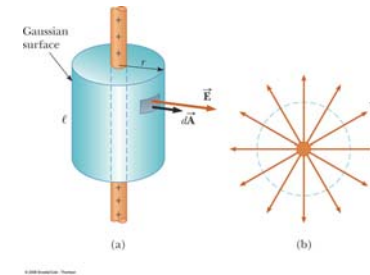
- For a plate of charge Q per unit area we define $\sigma = Q/A$, and chose the area vector to be in the direction of the electric field as shown right. Evaluating Gauss' Law we



$$\text{have } \oint_{\text{Surface}} \vec{E} \cdot d\vec{a} = E \oint_{\text{plate}} da = 2EA = \frac{Q_{\text{enclosed}}}{\epsilon_o} \rightarrow E_{\text{plate}} = \frac{\sigma A}{2A\epsilon_o} = \frac{\sigma}{2\epsilon_o}, \text{ which is constant}$$

and independent of how far you are away from the plate.

- For a line of charge Q per unit length we define $\lambda = Q/L$, and chose the area vector to be in the direction of the electric field which points radially out from the line of charge as shown below. Evaluating Gauss' Law we



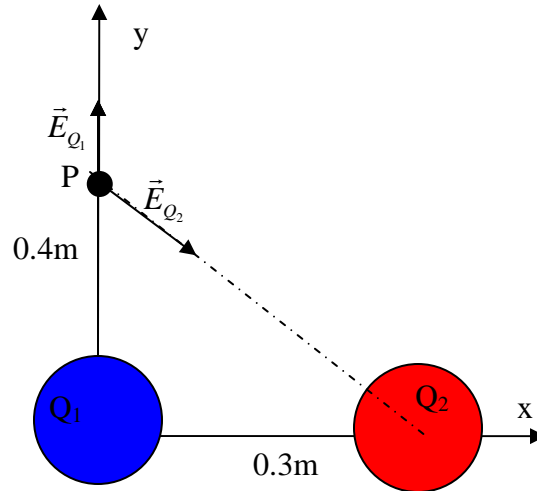
$$\text{have } \oint_{\text{Surface}} \vec{E} \cdot d\vec{a} = E \oint_{\text{line of charge}} da = 2\pi r l E = \frac{Q_{\text{enclosed}}}{\epsilon_o} \rightarrow E_{\text{line}} = \frac{\lambda l}{2\epsilon_o \pi r l} = 2k \frac{\lambda}{r}, \text{ which varies as}$$

the inverse of the distance from the line of charge.

Returning to point charges, we will try to evaluate the electric field at a point P in space given some different distributions of charge. To do this we will use the method outlined in the section on Coulomb's Law for solving vector problems.

Example #12 – What is the electric field at point P (located at a distance of 0.4m on the y-axis) due to the two charges ($Q_1 = 7 \mu\text{C}$ and $Q_2 = -5 \mu\text{C}$) located on the x-axis and separated by 0.3 m?

Solution – Given the diagram below, we draw the electric fields due to the two charges at point P.



Breaking these electric field vectors up into their x- and y-components we find

$$E_{net,x} = k \frac{Q_2}{r_{2,p}^2} \cos \theta = 9 \times 10^9 \frac{\text{Nm}^2}{\text{C}^2} \times \frac{5 \times 10^{-6} \text{C}}{(0.5\text{m})^2} \times \left(\frac{0.3\text{m}}{0.5\text{m}} \right) = 1.08 \times 10^5 \frac{\text{N}}{\text{C}}$$

$$E_{net,y} = +k \frac{Q_1}{r_{1,p}^2} - k \frac{Q_2}{r_{2,p}^2} \sin \theta = 9 \times 10^9 \frac{\text{Nm}^2}{\text{C}^2} \times \left[\frac{7 \times 10^{-6} \text{C}}{(0.4\text{m})^2} - \frac{5 \times 10^{-6} \text{C}}{(0.5\text{m})^2} \times \left(\frac{0.4\text{m}}{0.5\text{m}} \right) \right] = 2.50 \times 10^5 \frac{\text{N}}{\text{C}}$$

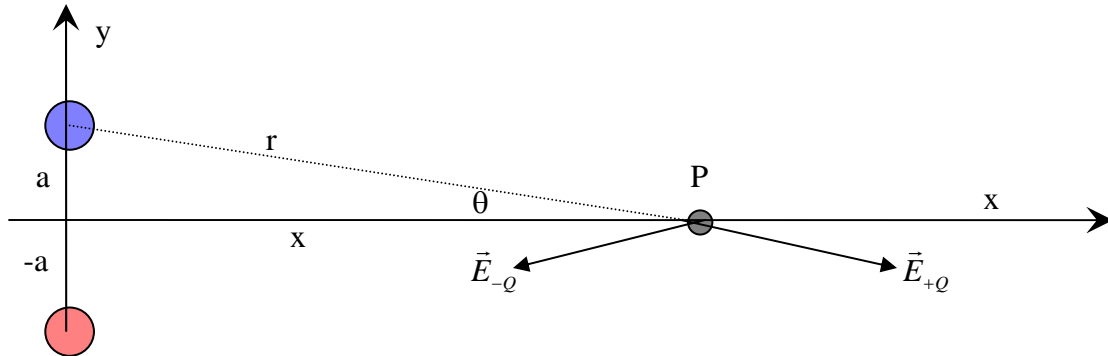
Using the Pythagorean Theorem we can express the net electric field at point P as

$$\vec{E}_{net} = \sqrt{\left(1.08 \times 10^5 \frac{\text{N}}{\text{C}}\right)^2 + \left(2.50 \times 10^5 \frac{\text{N}}{\text{C}}\right)^2} @ \phi = \tan^{-1} \left(\frac{2.50 \times 10^5 \frac{\text{N}}{\text{C}}}{1.08 \times 10^5 \frac{\text{N}}{\text{C}}} \right).$$

$$\vec{E}_{net} = 2.72 \times 10^5 \frac{\text{N}}{\text{C}} @ \phi = 79.4^\circ$$

Example #13 – What is the electric field at a point P (along the positive x-axis) due to the electric dipole situated on the y-axis.

Solution – First an electric dipole is constructed by two equal and opposite magnitude charges separated by a distance d between their centers. Here the distance between their centers is $d = 2a$, where a is the distance the charge is located above and below the x-axis. Second, we will need the distance between the charge and where we want to evaluate the electric field. This distance, r , is the same for both charges and is given as $r = \sqrt{a^2 + x^2}$. Third, given the coordinate system in the diagram, we draw the electric field at point P due to the two charges located on the y-axis. Then we break up the fields into their x- and y-components and calculate the net electric fields in the x- and y-directions.



In component form we find for the net electric fields in the x- and y-directions

$$E_{net,x} = +k \frac{Q}{r_{+Q,p}^2} \cos \theta - k \frac{Q}{r_{-Q,p}^2} \cos \theta = 0 \frac{N}{C}$$

$$E_{net,y} = -k \frac{Q}{r_{-Q,p}^2} \sin \theta - k \frac{Q}{r_{+Q,p}^2} \sin \theta = -2k \frac{Q}{r_{Q,p}^2} \sin \theta = -2k \frac{Qa}{(a^2 + x^2)^{\frac{3}{2}}}$$

This gives the net electric field in the negative y-direction with magnitude $2k \frac{Qa}{(a^2 + x^2)^{\frac{3}{2}}}$.

In fact this same analysis could be done anywhere along the y-axis and the result will be

the same magnitude and will point along the negative y-axis. If we take the limit that the point P is at a distance $x \gg a$, the dipole field reduces to $-2k \frac{Qa}{x^3} \approx -2k \frac{Qa}{r^3}$ and the field falls off as the inverse of the distance cubed!

In the above two examples we have been given a distribution of point charges and were asked to calculate the electric field at a point P in space. As a last couple of examples in this section, let's take a charge and see what happens as the charge experiences an electric field.

Example #14 – The electric field needed to balance a proton against gravity

What is the magnitude and direction of the electric field needed to balance the weight of a proton in the gravitational field of the earth?

Solution – The weight of the proton points vertically downward towards the earth and has a magnitude of mg . To balance the weight of the proton, the electric force must point vertically up, and since the proton carries a positive electric charge, the electric field must also point vertically up. Thus we have pointing vertically up, an electric field of magnitude

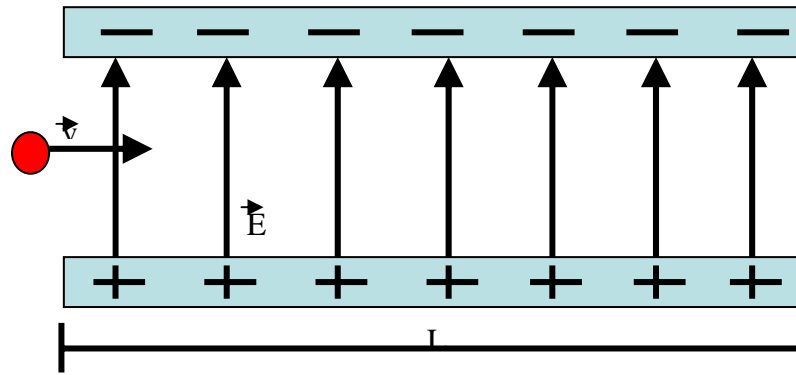
$$F_E - F_W = ma_y = 0 \rightarrow F_E = qE = mg \rightarrow E = \frac{mg}{q} = \frac{1.67 \times 10^{-27} \text{ kg} \times 9.8 \frac{\text{m}}{\text{s}^2}}{1.6 \times 10^{-19} \text{ C}} = 1.02 \times 10^{-7} \frac{\text{N}}{\text{C}}.$$

Example #15 – Projectile motion with an electron?

Suppose that an electron enters a region of uniform electric field ($E = 200 \text{ N/C}$) with an initial velocity of $3 \times 10^6 \text{ m/s}$. What are the acceleration of the electron, the time it takes

the electron to travel through the field and the vertical displacement of the electron while it is in the field if the length of the region $L = 0.1\text{m}$?

Solution – From the diagram of the set-up below, we see that a force will accelerate the electron vertically down since the electron has a negative charge. This acceleration will be constant, since we are told that the electric field is uniform, and this produces a constant force. Thus the acceleration of the electron is vertically down with



magnitude $a = \frac{F}{m} = \frac{qE}{m} = \frac{1.6 \times 10^{-19} \text{ C} \times 200 \frac{\text{N}}{\text{C}}}{9.11 \times 10^{-31} \text{ kg}} = 3.86 \times 10^{13} \frac{\text{m}}{\text{s}^2}$. This force only affects the

vertical motion of the electron and the horizontal component of the velocity is therefore a constant of the motion. The vertical component of the velocity will change with time.

The time it takes then, for the electron to traverse the length of the plates is given

as $t = \frac{L}{v_x} = \frac{0.1\text{m}}{3.0 \times 10^6 \frac{\text{m}}{\text{s}}} = 3.3 \times 10^{-8} \text{ s}$. In this time the electron is accelerated downward

toward the positive plate. Defining y_i to be zero where the electron enters, the vertical

displacement is $y_f = -\frac{1}{2} a_y t^2 = -\frac{1}{2} \times 3.51 \times 10^{13} \frac{\text{m}}{\text{s}^2} \times (3.3 \times 10^{-8} \text{ s})^2 = -1.95 \times 10^{-2} \text{ m}$ or 1.95cm

below where the electron entered.

Comments: Here we have used the equation of motion developed in the mechanics portion of the course valid for motion with constant acceleration. If we ignore the plates as the cause of the force on the electron and further pretend that the electron were not

charged, but simply a mass m moving under the influence of gravity, we have a case of two dimensional motion, or more specifically projectile motion, here with an electron in an electric field.

Example #16 – What was the work done on the electron in example #15 and does this equal the change in kinetic energy of the electron?

Solution – We will tackle this problem in two parts; first we will calculate the work done and then we will calculate the final velocity from the equations of motion and compute the change in kinetic energy for the electron in this case. So, the work done by the electrostatic force in accelerating the electron is given as

$$W_E = \vec{F} \cdot \Delta \vec{y} = F \Delta y \cos \theta = qE \Delta y = 1.6 \times 10^{-19} C \times 200 \frac{N}{C} \times 1.95 \times 10^{-2} m = 6.24 \times 10^{-19} J .$$

From Example 15, we know that the horizontal component of the electron's velocity is constant, so $v_{fx} = 3.0 \times 10^6 \frac{m}{s}$. The final vertical component of the electron's velocity is $v_{fy} = -a_y t = -3.51 \times 10^{13} \frac{m}{s^2} \times 3.3 \times 10^{-8} s = -1.17 \times 10^6 \frac{m}{s}$. Using the Pythagorean

Theorem, the final magnitude of the electron's velocity

$$\text{is } v_f = \sqrt{v_{fx}^2 + v_{fy}^2} = \sqrt{\left(3.0 \times 10^6 \frac{m}{s}\right)^2 + \left(-1.17 \times 10^6 \frac{m}{s}\right)^2} = 3.22 \times 10^6 \frac{m}{s} . \text{ To see if the work}$$

done equals the change in kinetic energy we need to compute the change in kinetic energy. The change in the electron's kinetic energy is

$$\Delta KE = \frac{1}{2} m (v_f^2 - v_i^2) = \frac{1}{2} \times 9.11 \times 10^{-31} kg \times \left(\left(3.22 \times 10^6 \frac{m}{s}\right)^2 - \left(3.0 \times 10^6 \frac{m}{s}\right)^2 \right) = 6.22 \times 10^{-19} J ,$$

which seems to agree pretty well. Thus, the work done is equal to the change in the electron's kinetic energy as it should by the work-kinetic energy theorem.

This last example brings us to our next topic, the work done in assembling or moving charges around in the electric fields of other charges. We are starting to bridge the space between having purely static charges and finally allowing these charges to move, thus establishing electric currents and the existence of magnetic fields and forces. In the next chapter we will examine the concept of the electric potential and of electric potential energy. We will start by looking at gravity and recalling what we know about how conservative forces give rise to conserved quantities like energy. Then we will, by analogy with gravity, discuss some fundamental concepts in our study of electricity.