Physics in Modern Medicine **Winter 2023**

Homework #6

Chapter 6 – Images from Radioactivity

Questions:

Q6.1 Alpha and beta particles have very little penetrating power in materials. If they were located deep within a structure they would never be able to get out to be detected. Gamma rays on the other hand have no problems with penetration and can easily escape the body.

Problems:

P6.1

a. For iodine-123, the half-life is 13 hours, so two days is 48 hours. This is somewhat less than 4 half-lives. We know the source activity after time t, so we can compute the initial source activity. From equation 6.1:

$$A = A_0 e^{-\lambda t} \rightarrow A_o = Ae^{\lambda t} = 2MBq \times e^{\frac{\ln 2}{13hrs}\times 48hrs} = 25.8MBq$$

Thus about $26MBq$ should initially be prepared to have only $2MBq$ remaining 2 days later.

b. The half-life of an iodine 131 source is 8.05 days. We can repeat the above calculation, and find:

$$A = A_0 e^{-\lambda t} \rightarrow A_o = Ae^{\lambda t} = 2MBq \times e^{\frac{\ln 2}{8.05day}\times 2day} = 2.4MBq$$

P6.2 Cobalt-60

a. The amount left after 11 years:

$A = A_0 e^{-\lambda t} = A_0 e^{-\frac{\ln 2}{5.3yrs}\times 11yrs} = 0.24A_0 \rightarrow 24\%$ of the initial source activity.

b. Assume here that the treatment time is 1 week, or $\frac{1}{52}$ of a year. The activity after one week would be:

$A = A_0 e^{-\lambda t} \rightarrow A_o = Ae^{\lambda t} = A_0 e^{-\frac{\ln 2}{5.3yrs}\times\frac{1}{52}yrs} = 0.997A_0$. After one week, 0.25%of the source activity has been lost. This is probably not something that would need to be corrected for during the treatment.

P6.3 The starting source activity of the technetium-99m is $100MBq$, and the desired source activity is $25MBq$, then since the half-life of technetium-99m is $6hrs$, we need to wait for a time:

$$A = A_0 e^{-\lambda t} \rightarrow t = -\frac{t_{1/2}}{\ln 2}\ln\left(\frac{A}{A_0}\right) = -\frac{6hrs}{\ln 2}\ln\left(\frac{25MBq}{100MBq}\right) = 12hrs$$

P6.6 Since the biological half-life is much longer than the physical half-life of the radioactive material, the effective half-life:

$\frac{1}{T_E} = \frac{1}{T_B} + \frac{1}{T_{1/2}} \rightarrow T_E \sim T_{1/2}$ and we need to determine the half-life of the substance to identify it.

We use the activity to determine the half-life:

$$A = A_0 e^{-\lambda t} \rightarrow t_{1/2} = -\frac{\ln(2) \times t}{\ln\left(\frac{A}{A_0}\right)} = -\frac{0.693 \times 3hrs}{\ln\left(\frac{59.3MBq}{185Bq}\right)} = 1.8hrs = 109.6min.$$

Looking this up in table 6.4, we find the element to be fluorine-18.

P6.9 PET Scanners

a. If the pair of coincidently detected gamma rays originate from a location which is x_1 away from one detector and x_2 away from the second detector, then we can use this information to generate the approximate annihilation location. Since the gamma rays travel at the speed of light, it takes a time $t_1 = \frac{x_1}{c}$ for a photon to cross distance x_2, and a time for a photon to cross distance $t_2 = \frac{x_2}{c}$. The difference in time recorded between the two detectors can be used to find a localization in space of the annihilation event.

$$\Delta t = t_2 - t_1 = \frac{x_2}{c} - \frac{x_1}{c} = \frac{x_2 - x_1}{c} = \frac{\Delta x}{c}.$$

b. Since the distance and time resolution are directly related, we can the equation above to find the distance resolution Δx from:

$$\Delta t = \frac{\Delta x}{c} \rightarrow \Delta x = c\Delta t = 3 \times 10^8 \frac{m}{s} \times 300 \times 10^{-12} s = 0.09m = 9cm.$$

To find the time resolution Δt from:

$$\Delta t = \frac{\Delta x}{c} = \frac{0.03m}{3 \times 10^8 \frac{m}{s}} = 1 \times 10^{-10} s = 0.1ns$$

Radioactivity Problems

1. Iodine ($^{131}_{53}I$) is used to clinically diagnose the thyroid gland. Suppose the patient is given an injection of $550\mu g$ of $^{131}_{53}I$. The half-life of $^{131}_{53}I$ is 8 days.
   a. What is the patient's activity immediately after injection?

   $550 \times 10^{-6} g\, ^{131}_{53}I \times \frac{1mol\, ^{131}_{53}I}{131g\, ^{131}_{53}I} \times \frac{6.23 \times 10^{23}}{1mol\, ^{131}_{53}I} = 2.5 \times 10^{18}\, ^{131}_{53}I$ atoms. The activity is the number of atoms times the decay constant and we have:

   $$A_0 = \lambda N_0 = \frac{\ln 2}{t_{1/2}} N_0 = \frac{\ln 2}{8day \times \frac{86400s}{1day}} \times 2.5 \times 10^{18} \times \frac{1Bq}{3.7 \times 10^{10} \frac{dec}{s}} = 67.7Bq$$

b. What is the patient's activity after 1 hour? After 10 hours?

$A = A_0 e^{-\lambda t} = 67.7Bqe^{-\frac{\ln 2}{8day}\times\left(1hr\times\frac{1day}{24hr}\right)} = 67.5Bq$ after one hour.

$A = A_0 e^{-\lambda t} = 67.7Bqe^{-\frac{\ln 2}{8day}\times\left(10hr\times\frac{1day}{24hr}\right)} = 65.3Bq$ after ten hours.

c. What is the patient's activity after 6 months?

$A = A_0 e^{-\lambda t} = 67.7Bqe^{-\frac{\ln 2}{8day}\times\left(6mo\times\frac{30day}{1mo}\right)} = 11.5\mu Bq \sim 0Bq$

d. Suppose that after initial injection of $550\mu g$, that 50% of it was excreted from the body after 48 hours. What is the remining activity in the patient's body on the third day of treatment if the biologic half-life of ${}^{131}_{53}I$ is $2days$? Hint, what is the effective half-life of the ${}^{131}_{53}I$?

$$\frac{1}{T_E} = \frac{1}{T_B} + \frac{1}{T_{1/2}} \rightarrow T_E = \left(\frac{1}{T_B} + \frac{1}{T_{1/2}}\right)^{-1} = \left(\frac{1}{2day} + \frac{1}{8day}\right)^{-1} = 1.6day.$$

What remains after three days

$= A_0 e^{-\lambda t} = 67.7Bqe^{-\frac{\ln 2}{1.6day}\times(3day)} = 18.5Bq$

Note: For parts a – c, we probably should have considered the biological half-life of the iodine as the body will excrete some of it. In these parts we did not.

2. A photomultiplier tube (PMT) is used for the detection of gamma radiation from a patient. The light that enters the tube interacts with a scintillator that changes the gamma-ray photons into visible light. The visible light is then detected by a barium oxide (BaO) cathode in the photomultiplier tube.

   a. Barium oxide has a work function of $1.3eV$. What wavelength should the gamma-ray photons have when they are emitted from the scintillator and to be detected by the strontium vanadate cathode?

   $$K = 0 = \frac{hc}{\lambda} - \phi$$
   $$\lambda = \frac{hc}{\phi} = \frac{6.63\times10^{-34}Js\times3\times10^{8}\frac{m}{s}}{1.3eV}\times\frac{1eV}{1.6\times10^{-19}J} = 9.6\times10^{-7}m$$
   $$\lambda = 960nm$$

   b. Suppose that one of these photons from part a strikes the cathode and liberates an electron. How high of a potential difference must the first barium oxide dynode be if $z = 2$ secondary electrons are liberated?

   $$W = -q\Delta V = N\phi \rightarrow \Delta V = \frac{N\phi}{-q} = \frac{2\times1.3eV}{e} = 2.6V$$

c. The PMT is made up of $N = 5$ dynodes in total. These are arranged in such a way that the same number of secondary electrons are liberated for each incident electron. How long does it take the signal to reach the anode if the separation between the dynodes is $d = 1cm$.

$W = -q\Delta V = Fd = mad$

$$a = \frac{-q\Delta V}{md} = \frac{2.6eV \times \frac{1.6\times10^{-19}J}{1eV}}{9.11\times10^{-31}kg \times 0.01m} = 4.6\times10^{13}\frac{m}{s^2}$$

$$d = \frac{1}{2}at^2 \rightarrow t_1 = \sqrt{\frac{2d}{a}} = \sqrt{\frac{2\times0.01m}{4.6\times10^{13}\frac{m}{s^2}}} = 2.1\times10^{-8}s$$

$t_{total} = Nt_1 = 1.04\times10^{-7}s$

d. How much current is measured at the anode if photons with a power $2\times10^{-8}W$ strike the cathode? When we say that we measure a current on our detector and this corresponds to a photon with a certain energy, this is how the detector electronics do the calculation.

$$P = \frac{dE}{dt} = \frac{d(E_\gamma N_\gamma)}{dt} = E_\gamma\frac{dN_\gamma}{dt} = \phi\frac{dN_\gamma}{dt}$$

$$N_e = z^N N_\gamma \rightarrow \frac{dN_e}{dt} = z^N\frac{dN_\gamma}{dt}$$

$$P = \phi z^{-N}\frac{dN_e}{dt} = \frac{\phi z^{-N}}{e}\frac{d(eN_e)}{dt} = \frac{\phi z^{-N}}{e}I \rightarrow I = \frac{eP}{\phi}z^N$$

$$I = \frac{1.6\times10^{-19}C \times 2\times10^{-8}W}{1.3eV\times\frac{1.6\times10^{-19}J}{1eV}}\times2^5 = 4.9\times10^{-7}A = 492nA$$

3. In PET, a radiopharmaceutical is introduced into an organism. The annihilation of an emitted positron and an electron from the surrounding tissue produces two photons that will be detected. We say that the positron gets ejected and quickly annihilates. What happens is we form a quasi-atom of the positron and electron called positronium.

a. Show that in the decay of such a quasi-atom, that if the speeds of the electron and positron are ignored, the two photons are emitted in opposite directions and determine the energy of a photon in eV.

To determine the energy of a photon emitted we use conservation of energy.

$\Delta E = E_f - E_i = 0 \rightarrow E_f = E_i \rightarrow 2E_{photon} = K_{e^-} + E_{rest,e^-} + K_{e^+} + E_{rest,e^+}$

If the electron and positron are "at rest" at the time of annihilation,

$K_{e^-} = K_{e^+} = 0$, and $2E_{photon} = 2E_{rest,e^-} \rightarrow E_{photon} = E_{rest,e^-} = mc^2 = 0.511MeV$.

To determine the directionality, we use conservation of momentum.

$\Delta p = p_f - p_i = 0 \rightarrow p_f = p_i = 0 \rightarrow 0 = p_{e^-} + p_{e^+} \rightarrow p_{e^-} = -p_{e^+}$

And the positron and electron are emitted $180^0$from each other.

b. What effects occur if the positronium quasi-atom has a kinetic energy, and what relevance does this mean for image production?

This is not easy to show, but if the e^-/e^+ was not at rest at the time of annihilation, then the photons will be produced at angles less than 180^0. This degrades location resolution on the image.

Since there is an initial speed, the signals could be doppler shifted as they propagate to the detector. This would introduce uncertainties in the localization of the annihilation event.