Chapter 7 – Momentum

We've talked the last few weeks about forces and how they relate to the change in an object's motion. We've also defined the motion of an object through its momentum, or the product of its mass and velocity. Symbolically we write this as $\vec{p} = m\vec{v}$ in units of $\frac{kg \cdot m}{s}$. Then the net force is time rate of change of the objects momentum, and we call these Newton's laws of motion.

$$\vec{F}_{net} = \frac{d\vec{p}}{dt} = m\vec{a}_{net} \qquad (1)$$

Consider an object of mass m. If net external force acting on the object is zero, then for any non-zero time interval $dt \neq 0$, equation 1 becomes

$$\vec{F}_{net} = 0 = \frac{d\vec{p}}{dt} \rightarrow d\vec{p} = 0 \rightarrow \vec{p}_f = \vec{p}_i \qquad (2)$$

and the objects momentum is a constant and the object travels in a straight line at a constant speed. Equation (2) is, of course, Newton's 1st law of motion.

We call equation (1) an interaction and the object has to interact with something in its environment in order for a force to be imparted on the object. Thus the object interacts with something in its environment and the interaction of the object with the environment exerts a force on the object. And by equation (1), this changes the momentum of the object. We call the time interval in equation (1) the interaction time. Thus we could write equation (1) as

$$d\vec{p} = \vec{p}_f - \vec{p}_i = \vec{F}_{net} dt \rightarrow \vec{p}_f = \vec{p}_i + \vec{F}_{net} dt \qquad (3).$$

Equation (3) is called the impulse-momentum theorem, where we define the impulse $\vec{I}$ given to an object as $\vec{I} = \vec{F}_{net} dt$. Equation (3) implies that the change in the momentum of the object is due to an interaction of the object with its environment for a time dt. By assumption, the interaction time $dt \neq 0$ and if $\vec{F}_{net} = 0$ then equation (3) is merely a restatement of Newton's 1st law of motion, equation (2). If $\vec{F}_{net} \neq 0$ then equation (1) is a measure of the interaction and we define the interaction to be equation (1), which we have called Newton's 2nd law of motion. Equation (1) gives us a way to quantify the interaction, or the change in motion and this change in motion of the mass m we have called the acceleration $\vec{a}_{net}$ of the system.

Suppose instead of a one mass system, that we have two objects of masses m_1 and m_2 interacting. That is, m_1 and m_2 are exerting forces on one another. At some time, let the momenta $\vec{p}_1$ and $\vec{p}_2$ of masses m_1 and m_2 respectively be given as shown in figure 1 below. Masses m_1 and m_2 are interacting and the forces involved in their interactions are given as $\vec{F}_{1,2}$ and $\vec{F}_{2,1}$. $\vec{F}_{1,2}$ is the force exerted on mass m_1 due to its interaction with mass m_2, while similarly $\vec{F}_{2,1}$ is the force exerted on m_2 due to its interaction with mass m_1. The object with mass m_1 experiences a change in its momentum given by equation (1) as $\vec{F}_{1,2} = \frac{d\vec{p}_1}{dt}$ due to its interaction with mass m_2.

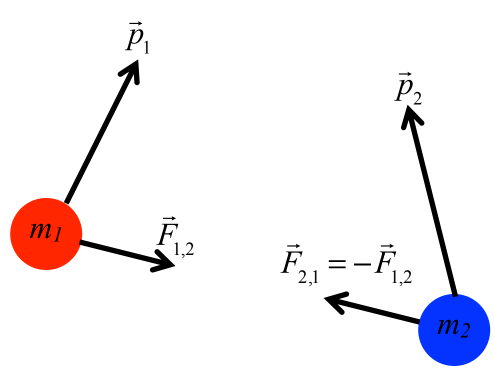

Figure 1: Two objects interacting, exerting equal and opposite forces on each other.

Analogously, the object with mass m_2 experiences a change in its momentum given by equation (1) as $\vec{F}_{2,1} = \frac{d\vec{p}_2}{dt}$ due to its interaction with mass m_1. By Newton's 3rd law of motion we have $\vec{F}_{1,2} = -\vec{F}_{2,1}$. Or, equivalently we can write $\vec{F}_{1,2} = -\vec{F}_{2,1} \rightarrow \frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt}$. This implies that the change in the momentum of the system of masses m_1 and m_2 does not change for any non-zero interaction time dt, which is of course the same for both objects interacting. Thus we can write $\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = \frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right) = \frac{d\vec{p}_{system}}{dt} = 0$. Therefore, using equation (1) we have that

$$d\vec{p}_{system} = \vec{p}_{f,system} - \vec{p}_{i,system} = 0 \rightarrow \vec{p}_{f,system} = \vec{p}_{i,system} \qquad (4).$$

Equation (4) states that the total momentum of the system is constant and does not change in time and thus the net force, by equation (1), that acts on the system must be zero. We call equation (4) a statement of conservation of momentum. Total momentum of the system is a conserved quantity and the total momentum of the system does not change in during the interaction. The individual momenta of each of the masses may change due the interaction, but the entire momentum of the system is constant. We have to be careful of how we define our system.

Example 1: Suppose that a ball of mass $m = 200g$ is thrown at a wall at an angle of $\theta = 60^0$, measured with respect to the vertical, as shown in Figure 2. What are the components of the changes in the ball's momentum, Δp_x and Δp_y respectively? What is the change in the ball's momentum, $\Delta\vec{p}$, if the ball has an initial speed of $v_i = 5\frac{m}{s}$ and a final speed approximately equal to that of the initial speed. If the ball is in contact with the wall for a time of $\Delta t = 0.2s$, what force does the wall exert on the ball? What force does the ball exert on the wall?

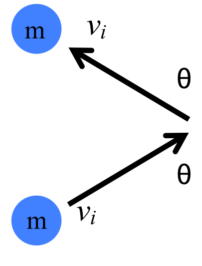

Figure 2: A ball is thrown off at a wall and bounces off at approximately the same speed as it was incident.

Solution: The change in the components of the momentum for the ball, assuming that the wall doesn't move is given as

$$\Delta p_x = p_{fx} - p_{ix} = -mv_{ix} - mv_{ix} = -2mv_i \sin\theta = -2\times 0.2kg \times 5\tfrac{m}{s} \times \sin 60 = -1.73\tfrac{kg\cdot m}{s}$$
$$\Delta p_y = p_{fy} - p_{iy} = mv_{fy} - mv_{iy} = mv_i \cos\theta - mv_i \cos\theta = 0\tfrac{kg\cdot m}{s}$$

Thus the change in momentum of the ball is

$$\Delta\vec{p} = \sqrt{\Delta p_x^2 + \Delta p_y^2} \ @\phi = \tan^{-1}\left(\frac{\Delta p_y}{\Delta p_x}\right)$$
$$\Delta\vec{p} = \sqrt{\left(1.73\tfrac{kg\cdot m}{s}\right)^2 + \left(0\tfrac{kg\cdot m}{s}\right)^2} \ @\phi = \tan^{-1}\left(\frac{0\tfrac{kg\cdot m}{s}}{-1.73\tfrac{kg\cdot m}{s}}\right)$$
$$\Delta\vec{p} = 1.73\tfrac{kg\cdot m}{s} \ @\phi = 180^0$$

The ball experiences a change in momentum horizontally but continues to move vertically upward.

The wall exerts a force on the ball given by

$$\vec{F}_{ball,wall} = \frac{\Delta\vec{p}}{\Delta t} = \frac{1.73\tfrac{kg\cdot m}{s}}{0.2s} @\phi = 180^0 = 8.65N @\phi = 180^0 .$$

By Newton's third law, the ball exerts a force of equal magnitude but opposite direction on the wall. Thus $\vec{F}_{wall,ball} = 8.65N @\phi = 0^0$.

Example 2: Consider the following system, shown in Figure 3, in which two blocks of masses m and $3m$ respectively are placed on a horizontal, frictionless surface. A light (i.e. massless) spring is attached to one block and the blocks are squeezed together and tied by a light cord. If the cord is cut and the block of mass $3m$ moves to the right with a speed of a $v_{3m} = 2\tfrac{m}{s}$, what is the speed of the block of mass m?

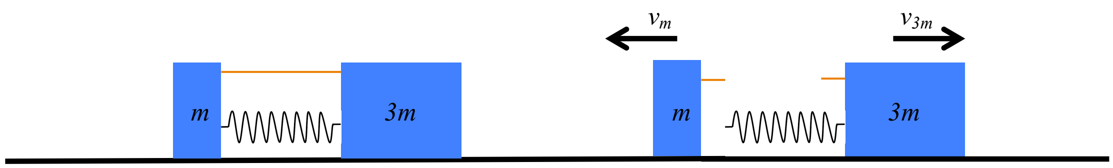

Figure 3: Two masses on a horizontal surface for Example 2.

Solution: Since the momentum of the system is conserved we have, from equation (4), $\Delta p_{system} = p_f - p_i = 0$, with $p_i = 0$. Thus we can write our statement of conservation of momentum as:

$$\Delta p_{system} = p_f - p_i = 0 \rightarrow p_f = mv_m + (3m)v_{3m} = 0$$

$$\therefore v_m = -\frac{(3m)}{m}v_{3m} = -3v_{3m} = -3\times 2\tfrac{m}{s} = -6\tfrac{m}{s}$$

Therefore the block of mass m moves to the left (as expected if the block of mass $3m$ moves right) at a speed of $6\tfrac{m}{s}$.

Example 3: In Example 2, how much energy was initially stored in the compressed spring, if the mass of the smaller block is $m = 0.35kg$?

Solution: If we take the system as the two masses, the spring, and the world, then change in energy of the system is zero. We have

$$\Delta E_{system} = 0 = \Delta K_m + \Delta K_{3m} + \Delta U_g + \Delta U_s = \left(\tfrac{1}{2}mv_m^2 - 0\right) + \left(\tfrac{1}{2}(3m)v_{3m}^2 - 0\right) + \left(U_{S,f} - U_{S,i}\right)$$

$$\therefore U_{S,i} = \tfrac{1}{2}mv_m^2 + \tfrac{1}{2}(3m)v_{3m}^2 = \tfrac{1}{2}(0.35kg)\left[\left(-6\tfrac{m}{s}\right)^2 + 3\left(2\tfrac{m}{s}\right)^2\right] = 8.4J$$

Collisions

Returning to Figure (1) we have two objects coming together, interacting and then perhaps moving apart again. Whether the objects make physical contact or not, this is an example of a collision. To begin our study collisions between two objects let's take two objects and make them interact (collide) in one dimension. Consider Figure 4 below, which shows an object of mass m_1 moving with a velocity v_{1i}. In order to make the problem less algebraically intensive, let us take the mass m_2 to be initially at rest. Of course, mass m_2 could be moving in the same direction as mass m_1 or mass m_2 could be moving directly at mass m_1. We'll worry about that in a bit. Our goal here is to determine the velocities of the both masses after the collision. This of course depends on the type of collision.

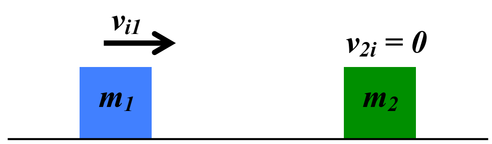

Figure 4: Mass m_1 moves to the right at an initial velocity v_{1i} and collides with mass m_2 initially at rest.

There are two types of collisions that we will investigate and they are called inelastic and elastic. No matter which type of collision we investigate, if the collision time is small and we assume that there are no external forces acting on the masses during the collision, then momentum is conserved, as is given by equation (4). What separates the collision types is whether the energy due to the motion of the objects, that is the kinetic energy, is conserved or not. In general, the total kinetic energy of the system of objects is not a conserved quantity since there are energy losses to sound (you can hear the objects collide) and deforming the objects (think cars crumpling when they collide) but we can approximate situations in which the kinetic energy is conserved.

Inelastic Collisions

Inelastic collisions are those in which the momentum is conserved but the kinetic energy is not. Total energy is always conserved. Consider the following situations in which we have two objects colliding. Returning to Figure 4 above suppose that the two objects stick together after the collision, as shown in Figure 5 below. We would like to determine the velocity of the system after the collision? To determine the velocity we apply conservation of momentum, and assuming that to the right is the positive x-direction, we have

$$\Delta p_x = p_{fx} - p_{ix} = 0$$
$$\therefore p_{ix} = p_{fx} \rightarrow m_1 v_{i1} = m_1 v_{1f} + m_2 v_{2f} = \left(m_1 + m_2\right) V$$

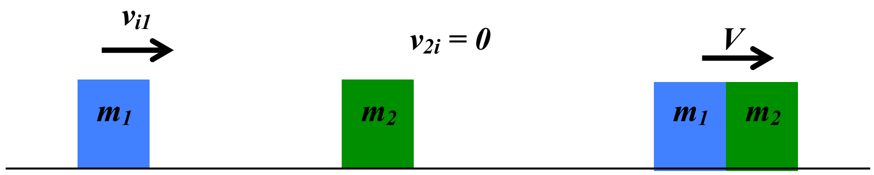

Figure 5: Mass m_1 moves to the right at an initial velocity v_{1i} and collides with mass m_2 initially at rest. After the collision, masses m_1 and m_2 move off together with a common velocity V.

Solving for the velocity of the system after the collision we get $V = \left(\frac{m_1}{m_1 + m_2}\right)v_{1i}$. If $m_1 \gg m_2$ then the velocity of the system after the collision V is approximately equal to the velocity of m_1 before the collision, v_{1i}. If $m_1 \ll m_2$ then the velocity of the system after the collision V is very small compared to the velocity of m_1 before the collision, v_{1i}, but it is not zero. Next, let us calculate the kinetic energy before and after the collision and then the change in kinetic energy. If kinetic energy is conserved, then $\Delta K = 0$; otherwise it is not. The kinetic energy before the collision is $K_i = \frac{1}{2}m_1 v_{1i}^2$. The kinetic energy after the collision is

$K_f = \frac{1}{2}\left(m_1 + m_2\right)V^2 = \frac{1}{2}\left(m_1 + m_2\right)\left(\frac{m_1^2}{\left(m_1 + m_2\right)^2}\right)v_{i1}^2 = \frac{1}{2}\left(\frac{m_1^2}{m_1 + m_2}\right)v_{i1}^2$. Taking the difference, which I'm not going to physically write out, we get that the change in kinetic energy is not equal to zero and thus kinetic energy is not conserved. The difference between K_f and K_i is the energy lost to the collision, as heat, deforming the objects, and sound.

Before we move on to elastic collisions let's do one more example, but with some numbers. Again, as in Figure 4, let the cars collide but this time let them not stick together after the collision, but rather move separately but in the same direction, as seen in Figure 6. Let the cars have masses $m_1 = 1200kg$ and $m_2 = 9000kg$ with velocities $v_{1i} = 25\frac{m}{s}$ and $v_{2i} = 20\frac{m}{s}$, initially, respectively. After the collision let the velocity of car 1 be $v_{2f} = 18\frac{m}{s}$ while the velocity of car

2 is unknown, call it v_{2f}. Let's determine the velocity of car 2 by applying conservation of momentum.

$$\Delta p_x = p_{fx} - p_{ix} = 0$$
$$\therefore p_{ix} = p_{fx} \rightarrow m_1 v_{i1} + m_2 v_{21} = m_1 v_{1f} + m_2 v_{2f}$$
$$v_{2f} = \frac{m_1 v_{i1} + m_2 v_{21} - m_1 v_{1f}}{m_2} = \frac{\left(1200kg \times 25\tfrac{m}{s}\right) + \left(9000kg \times 20\tfrac{m}{s}\right) - \left(1200kg \times 18\tfrac{m}{s}\right)}{9000kg} = 20.9\tfrac{m}{s}$$

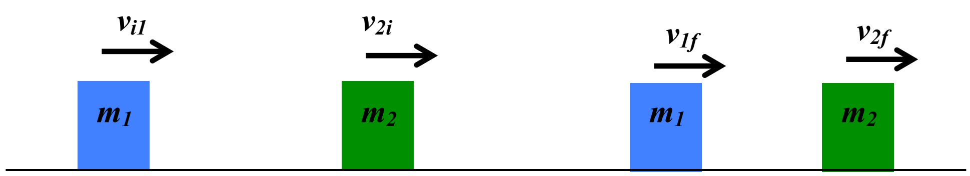

Figure 6: Mass m_1 moves to the right at an initial velocity v_{1i} and collides with mass m_2 moving to the right with velocity v_{2i}. After the collision, masses m_1 and m_2 move off separately with final velocities v_{1f} and v_{2f}.

If this is an inelastic collision then the change in kinetic energy should be zero. To see whether the change in kinetic energy is zero or not, calculate the initial and final kinetics energies before and after the collision. The initial kinetic energy is

$K_i = \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} \times 1200kg \times \left(25\tfrac{m}{s}\right)^2 + \frac{1}{2} \times 9000kg \times \left(20\tfrac{m}{s}\right)^2 = 2.18 \times 10^6 J$, while the final kinetic energy,

$K_f = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 = \frac{1}{2} \times 1200kg \times \left(18\tfrac{m}{s}\right)^2 + \frac{1}{2} \times 9000kg \times \left(20.9\tfrac{m}{s}\right)^2 = 2.16 \times 10^5 J$. The change in kinetic energy is $\Delta K = K_f - K_i = \left(2.18 - 2.16\right) \times 10^6 J = 20000J$ is lost to the collision. This is an inelastic collision, but not by that much.

Elastic Collisions

Elastic collisions conserve both momentum and kinetic energy. For macroscopically sized objects, completely elastic collisions are an approximation. Consider Figure 7 below in which mass m_1 is moving to the right at an initial speed v_{1i}, while mass m_2, located to the right of mass m_1, is initially at rest. Mass m_1 collides elastically with mass m_2 and we would like to calculate the final velocities of each of the masses after the collision, v_{1f} and v_{2f}. Assuming that no external forces act during the collision, we apply conservation of momentum and kinetic energy to determine the two unknown velocities.

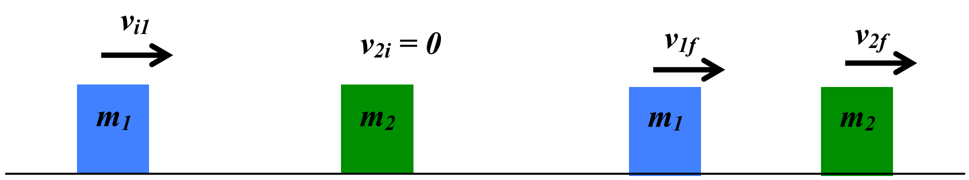

Figure 7: Mass m_1 moves to the right at an initial velocity v_{1i} and collides with mass m_2 at rest. After the collision, masses m_1 and m_2 move off separately with final velocities v_{1f} and v_{2f}.

Here we assume that both masses are moving to the right after the collision. In particular, mass m_2 will most likely move to the right after the collision while mass m_1 may move to the right or it could move to the left. We'll determine the actual directions by solving the equations for momentum and kinetic energy. Applying conservation of momentum and kinetic energy we get

$$\Delta p_{system} = 0 \rightarrow p_{i,system} = p_{f,system} \rightarrow m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\Delta K = 0 \rightarrow K_{i,system} = K_{f,system} \rightarrow \tfrac{1}{2} m_1 v_{1i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2$$

We have two equations and two unknowns. From the equation for momentum, we solve for the velocity of mass m_2 after the collision v_{2f} and substitute this into the equation for kinetic energy. Thus $v_{2f} = \frac{m_1}{m_2}\left(v_{1i} - v_{1f}\right)$ and $\tfrac{1}{2} m_1 v_{1i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 \left(\frac{m_1^2}{m_2^2}\right)\left(v_{1i} - v_{1f}\right)^2$.

After some algebra we can determine expressions for the final velocities of mass m_1 and m_2 after the collision. We find

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} \qquad (5a)$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} \qquad (5b)$$

Let's check some limiting cases of the masses to see if equations 5a and 5b seem reasonable. Suppose that $m_1 \ll m_2$. Equations (5a) and (5b) become

$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} \sim -v_{1i}$ and $v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} \sim 0$ respectively. Here the lighter mass m_1 bounces off of the heavier mass m_2 in the opposite direction with very little loss in speed and mass m_2 remains stationary. If $m_1 \gg m_2$, equations (5a) and (5b) become $v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} \sim v_{1i}$ and $v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} \sim 2v_{1i}$. Here the heavier mass m_1 keeps going in the same direction with little loss of speed while the lighter mass m_2 gets a large kick in speed. If the two masses are approximately equal $m_1 \sim m_2$, equations (5a) and (5b) become $v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} \sim 0$ and $v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} \sim v_{1i}$. Here the incident mass m_1 comes to rest and mass m_2 leaves with the speed of the incident mass. Equations (5a) and (5b) therefore seem reasonable.

Example 4: Consider the frictionless track ABC as shown below in Figure 8. A block of mass $m_1 = 5kg$ is released from point A. It makes a head-on collision at point B with a block of mass $m_2 = 10kg$, initially at rest. What is the maximum height to which mass m_1 rises back up the track after the collision? In addition, suppose that just past point C there is a rough region in which the coefficient of friction is $\mu = 0.9$. After how much distance would block come to rest?

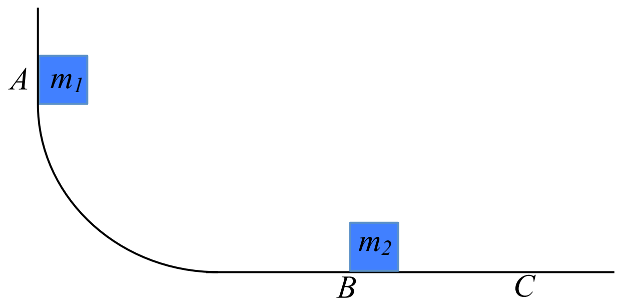

Figure 8: Two masses on a frictionless track.

Solution: Assuming mass m_1 and the world are the system, energy is conserved.

Applying conservation of energy between points A and B (just before m_1 collides with m_2) we have

$$\Delta E = \Delta K_1 + \Delta U_{g1} + \Delta U_s = 0$$

$$\rightarrow \left(\tfrac{1}{2}m_1 v_{1f}^2 - 0\right) + \left(0 - m_1 g y_{i1}\right) + \left(0 - 0\right) = 0$$

$$v_{1f} = \sqrt{2gy_{1i}} = \sqrt{2 \times 9.8\tfrac{m}{s^2} \times 5m} = 9.9\tfrac{m}{s}$$

During the collision between blocks m_1 and m_2 at point B, total momentum of the system is conserved and if we model this as a completely elastic collision, kinetic energy is also conserved. Applying conservation of momentum and kinetic energy during the collision we have

$$\Delta p_{system} = 0 \rightarrow p_{i,system} = p_{f,system} \rightarrow m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\Delta K = 0 \rightarrow K_{i,system} = K_{f,system} \rightarrow \tfrac{1}{2}m_1 v_{1i}^2 = \tfrac{1}{2}m_1 v_{1f}^2 + \tfrac{1}{2}m_2 v_{2f}^2$$

The solutions are given by equations 5a and 5b. We have

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} = \left(\frac{5kg - 10kg}{5kg + 10kg}\right) \times 9.9\tfrac{m}{s} = -3.3\tfrac{m}{s}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} = \left(\frac{2 \times 5kg}{5kg + 10kg}\right) \times 9.9\tfrac{m}{s} = 6.5\tfrac{m}{s}$$

To determine the height mass m_1 rises back up the track after the collision we apply conservation of energy and we have

$$\Delta E = \Delta K_1 + \Delta U_{g1} + \Delta U_s = 0$$

$$\rightarrow \left(0 - \tfrac{1}{2} m_1 v_{1i}^2\right) + \left(m_1 g y_{f1} - 0\right) + \left(0 - 0\right) = 0$$

$$y_{f1} = \frac{v_{1i}^2}{2g} = \frac{\left(-3.3 \tfrac{m}{s}\right)^2}{2 \times 9.8 \tfrac{m}{s^2}} = 0.56m$$

Mass m_2 moves to the right after the collision and when it encounters the rough surface, friction does work on mass bringing it to rest. The distance traveled by mass m_2 is given by

$$W_{fr} = \Delta K \rightarrow F_{fr} \Delta x \cos\theta = \mu F_N \Delta x \cos\theta = \tfrac{1}{2} m_2 v_{2f}^2 - \tfrac{1}{2} m_2 v_{2i}^2$$

$$\Delta x = \frac{v_{2i}^2}{2\mu g} = \frac{\left(6.5 \tfrac{m}{s}\right)^2}{2 \times 0.9 \times 9.8 \tfrac{m}{s^2}} = 2.4m$$

Collisions in two-dimensions

The development of equations (1) – (4) apply whether the motion is in one or more than one dimension. Consider the arrangement of masses shown in Figure 9. Let mass m_1 be incident along the x-axis with velocity v_{1i} and let mass m_1 make a glancing collision with mass m_2 initially at rest. Take the origin of the coordinate system be at mass m_2. After the collision mass m_1 scatters at an angle θ measured with respect to the x-axis while mass m_2 scatters at angle ϕ, also measured with respect to the x-axis.

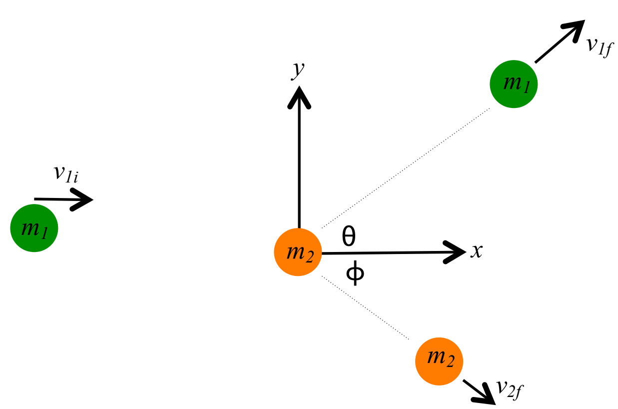

Figure 9: Two masses undergo a glancing collision in two-dimensions.

We can see where mass m_1 goes after the collision, so we assume that we know of can measure θ. We would like to calculate the velocities of each of the masses after the collision (v_{1f} and v_{2f}) and the scattering angle of mass m_2, ϕ. Assume the collision is elastic and we conserve momentum in the x- and y-directions as well as kinetic energy. Conservation of momentum horizontally and vertically gives us

$$x:\ \ m_1 v_{1i} = m_1 v_{1f}\cos\theta + m_2 v_{2f}\cos\phi \quad (6a)$$
$$y:\ \ 0 = m_1 v_{1f}\sin\theta - m_2 v_{2f}\sin\phi \quad (6b)$$

while conservation of kinetic energy gives

$$\tfrac{1}{2}m_1 v_{1i}^2 = \tfrac{1}{2}m_1 v_{1f}^2 + \tfrac{1}{2}m_2 v_{2f}^2 \quad (7).$$

In theory the above three equations (6a), (6b), and (7) are solvable for the three unknowns that we have. However, the algebra can be challenging. Let's do this by using some numbers. First, let $m_1 = m_2 = m$ and $v_{1i} = 3\frac{m}{s}$. What are the final velocities of each mass after the collision and at what angle ϕ does mass m_2 scatter if mass m_1 scatters at $\theta = 35^0$? Conservation of momentum horizontally and vertically becomes

$$x:\ \ v_{1i} = v_{1f}\cos\theta + v_{2f}\cos\phi$$
$$y:\ \ 0 = v_{1f}\sin\theta - v_{2f}\sin\phi$$

and conservation of kinetic energy is given as

$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2.$$

To solve we first square the x- and y-momentum equations, add the results together and then use conservation of kinetic energy.

$$x: \; v_{1i}^2 = v_{1f}^2\cos^2\theta + v_{2f}^2\cos^2\phi + 2v_{1f}v_{2f}\cos\theta\cos\phi$$
$$y: \; 0 = v_{1f}^2\sin^2\theta + v_{2f}^2\sin^2\phi - 2v_{1f}v_{2f}\sin\theta\sin\phi$$

$$\rightarrow v_{1i}^2 = v_{1f}^2\left(\cos^2\theta + \sin^2\theta\right) + v_{2f}^2\left(\cos^2\phi + \sin^2\phi\right) + 2v_{1f}v_{2f}\left(\cos\theta\cos\phi - \sin\theta\sin\phi\right)$$

Using conservation of energy we get, noting that $\sin^2\alpha + \cos^2\alpha = 1$

$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2 = v_{1f}^2 + v_{2f}^2 + 2v_{1f}v_{2f}\left(\cos\theta\cos\phi - \sin\theta\sin\phi\right).$$

The expression in parentheses is a trigonometric identity,
$\cos\theta\cos\phi - \sin\theta\sin\phi = \cos\left(\theta + \phi\right)$.

Now we can determine the scattering angle ϕ of m_2. We have, assuming that $v_{1f} \neq 0$ and $v_{2f} \neq 0$

$$0 = 2v_{1f}v_{2f}\cos\left(\theta+\phi\right) \rightarrow \cos\left(\theta+\phi\right) = 0 \rightarrow \theta + \phi = 35^0 + \phi = 90^0$$
$$\therefore \phi = 55^0$$

Here we note that ***IF and ONLY IF*** $m_1 = m_2$ the masses scatter at right angles to each other. If the masses are not equal then we have to go back equations (6a), (6b), and (7) and solve the problem with the unequal masses. Now that we have the scattering angle we can use equations (6a) and (6b) to solve for the final speeds of each mass after the collision. From equation (6b) we solve, say, for v_{2f}, and get

$v_{2f} = \left(\dfrac{\sin\theta}{\sin\phi}\right)v_{1f} = \left(\dfrac{\sin 35}{\sin 55}\right)v_{1f} = 0.7v_{1f}$. Inserting this into equation (6a) we can determine v_{1f}. We have

$v_{1i} = v_{1f}\cos 35 + \left(0.7v_{1f}\right)\cos 55 = 1.22v_{1f} \rightarrow v_{1f} = \dfrac{v_{1i}}{1.22} = \dfrac{3\frac{m}{s}}{1.22} = 2.46\frac{m}{s}$. And lastly

$v_{2f} = 0.7v_{1f} = 0.7 \times 2.46\frac{m}{s} = 1.72\frac{m}{s}$.

Of course we've also looked at inelastic collisions in which momentum is conserved, but not kinetic energy. How about we look at an inelastic collision in two dimensions. Consider the traffic accident shown in Figure 10 below in which two vehicles, a car (in blue) and a van (in green) approach an intersection. Both fail to stop and the cars collide, stick together and move off with a common final velocity. What is the final velocity of the cars after the collision?

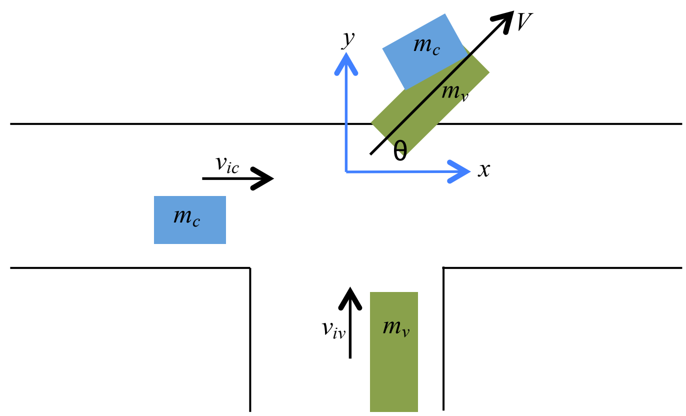

Figure 10: Inelastic collision of two vehicles at an intersection. The vehicles move off together after the collision.

Let's take up the page as the positive y-direction and to the right as the positive x-direction. Applying conservation of momentum horizontally and vertically we have

$$\begin{aligned} x: &\quad m_c v_{ic} = (m_c + m_v) V \cos\theta \\ y: &\quad m_v v_{iv} = (m_c + m_v) V \sin\theta \end{aligned}.$$

Dividing these two expressions will allow us to determine the angle θ that the velocity vector makes with the horizontal. Once we know the angle we can use either momentum equation to determine the final speed V of the car and van after the collision. The angle is determined as

$$\frac{m_v v_{iv}}{m_c v_{ic}} = \frac{(m_c + m_v)V\sin\theta}{(m_c + m_v)V\cos\theta} = \tan\theta$$

$$\tan\theta = \frac{m_v v_{iv}}{m_c v_{ic}} = \frac{2500kg \times 20\frac{m}{s}}{1500kg \times 25\frac{m}{s}} = 1.33 \rightarrow \theta = 53.1^0$$

The final speed of the car and van after the collision is

$$m_c v_{ic} = (m_c + m_v)V\cos\theta \rightarrow V = \frac{m_c v_{ic}}{(m_c + m_v)\cos\theta} = \frac{1500kg \times 25\frac{m}{s}}{(1500kg + 2500kg)\cos 53.1} = 15.6\frac{m}{s}$$

Summary

We've looked at momentum and its conservation in one and two dimension. We've also looked at two types of collisions between objects, namely inelastic and elastic. Elastic collisions conserve momentum and kinetic energy. Inelastic collisions conserve momentum only, with some of the energy of motion being lost to sources like heat, light, sound and deforming the objects. Both collisions conserve total energy. Unless explicitly stated, in order to truly tell if a collision is elastic or not, one needs to calculate the change in kinetic energy. If the change is zero, the collision is elastic. If the change is not zero then the collision is inelastic.

Let's try a few more examples for practice.

Example 5: Rutherford Backscattering Spectroscopy

> Alpha particles are routinely accelerated using a particle accelerator and are directed with the use of magnets into targets composed of various elements. A famous experiment called Rutherford's experiment has a beam of alpha particles incident on a target of gold. An alpha particle (a helium nucleus) is accelerated to a certain speed and makes an elastic head-on collision with a stationary gold nucleus. What percentage of its original kinetic energy is transferred to the gold nucleus?

Solution:

The percent kinetic energy lost is given by

$$\%_{lost} = \left[1 - \frac{KE_{after}}{KE_{initial}}\right] \times 100\% = \left[1 - \frac{\frac{1}{2} m_\alpha v^2_{\alpha,\text{after collision}} + \frac{1}{2} m v^2_{\text{Au,after collision}}}{\frac{1}{2} m v^2_{i,\alpha}}\right] \times 100\%$$

We need to determine both the final speed of the alpha particle and the gold nucleus after the collision. To do this we apply conservation of momentum and kinetic energy. From conservation of momentum we have $m_\alpha v_{i,\alpha} = m_\alpha v_{f,\alpha} + m_{Au} v_{f,Au}$ and from conservation of kinetic energy $\frac{1}{2} m_\alpha v^2_{i,\alpha} = \frac{1}{2} m_\alpha v^2_{f,\alpha} + \frac{1}{2} m_{Au} v^2_{f,Au}$.

Here we have two equations and two unknowns. From momentum we solve for the final velocity of the alpha particle and obtain $v_{f,\alpha} = v_{i,\alpha} - \frac{m_{Au}}{m_\alpha} v_{f,Au}$.

We square this result and substitute into the equation for kinetic energy we obtain a quadratic equation in the final velocity of the gold nucleus. The quadratic equation is $0 = \left(\frac{m^2_{Au}}{m_\alpha} + m_{Au}\right) v^2_{f,Au} - \left(2 m_{Au} v_{i,\alpha}\right) v_{f,Au}$. Using the quadratic formula we find the solutions $v_{f,Au} = \left\{ \begin{matrix} \left(\frac{2m_\alpha}{m_\alpha + m_{Au}}\right) v_{i,\alpha} \\ 0 \end{matrix} \right\}$ and reject the zero speed solution. Substituting this result into our equation for the final speed of the alpha particle and we calculate the final speed to be $v_{f,\alpha} = v_{i,\alpha} - \frac{m_{Au}}{m_\alpha} v_{f,Au} = \left(\frac{m_\alpha - m_{Au}}{m_\alpha + m_{Au}}\right) v_{i,\alpha}$. So the percent of the initial kinetic energy lost is

$$\%_{lost} = \left[1 - \frac{KE_{after}}{KE_{initial}}\right] \times 100\% = \left[1 - \frac{\frac{1}{2} m_\alpha v^2_{\alpha,\text{after collision}} + \frac{1}{2} m v^2_{\text{Au,after collision}}}{\frac{1}{2} m v^2_{i,\alpha}}\right] \times 100\%$$

$$\%_{lost} = \left[1 - \frac{\frac{1}{2} m_\alpha \left(\frac{m_\alpha - m_{Au}}{m_\alpha + m_{Au}}\right)^2 v^2_{i,\alpha} + \frac{1}{2} m_{Au} \left(\frac{2m_\alpha}{m_\alpha + m_{Au}}\right)^2 v^2_{i,\alpha}}{\frac{1}{2} m v^2_{i,\alpha}}\right] \times 100\%$$

$$\%_{lost} = \left[1 - \left(\frac{m_\alpha - m_{Au}}{m_\alpha + m_{Au}}\right)^2 + m_{Au}\left(\frac{2}{m_\alpha + m_{Au}}\right)^2\right] \times 100\%$$

Using the mass of an alpha particle of *4u* and of gold *197u*, we have

$$\%_{lost} = \left[1 - \left(\frac{m_\alpha - m_{Au}}{m_\alpha + m_{Au}}\right)^2 + m_{Au}\left(\frac{2}{m_\alpha + m_{Au}}\right)^2\right] \times 100\%$$

$$= \left[1 - \left(\frac{4-197}{4+197}\right)^2 + 197\left(\frac{2}{4+197}\right)^2\right] \times 100\% = 9.8\%$$

Example 6: Ice Hockey

A hockey puck traveling at $1.2\,\frac{m}{s}$ collides with a second stationary equal mass puck and, after the collision, moves with a speed of $0.8\,\frac{m}{s}$ deflected by an angle of 30^0. What is the velocity (magnitude and direction) of the other puck after the collision? In addition, what is the fraction of the initial energy lost in the collision?

Solution: Using a standard Cartesian coordinate system we apply conservation of momentum in the horizontal and vertical directions and we have $p_{ix} = p_{fx} \rightarrow mv_{1ix} = mv_1 \cos\theta + mv_2 \cos\phi \rightarrow v_{1ix} = v_1 \cos\theta + v_2 \cos\phi$ and $p_{iy} = p_{fy} \rightarrow 0 = mv_1 \sin\theta - mv_2 \sin\phi \rightarrow 0 = v_1 \sin\theta - v_2 \sin\phi$.

Here we have two equations and two unknowns, v_2 and ϕ. Inserting the numbers from the problem we have for the horizontal and vertical directions $v_{1ix} = v_1 \cos\theta + v_2 \cos\phi \rightarrow v_2 \cos\phi = 0.51$ and $v_2 \sin\phi = 0.4$.

Dividing these two expressions we solve for the unknown angle and find $\tan\phi = \dfrac{0.4}{0.51} = 0.7843 \rightarrow \phi = \tan^{-1}(0.7843) = 38.1^o$. Therefore the unknown velocity is $v_2 \sin\phi = v_2 \sin 38.1 = 0.4 \rightarrow v_2 = 0.65\,\frac{m}{s}$.

The fraction of the initial energy lost is

$$\%_{lost} = \left[1 - \frac{KE_{after}}{KE_{initial}}\right] \times 100\% = \left[1 - \frac{\frac{1}{2}mv_{1,\text{after collision}}^2 + \frac{1}{2}mv_{2,\text{after collision}}^2}{\frac{1}{2}mv_{1ix}^2}\right] \times 100\% = \left[1 - \frac{v_{1,\text{after collision}}^2 + v_{2,\text{after collision}}^2}{v_{1ix}^2}\right] \times 100\%$$

$$\%_{lost} = \left[1 - \frac{m}{m+M}\right] \times 100\% = \left[1 - \frac{(0.8\,\frac{m}{s})^2 + (0.65\,\frac{m}{s})^2}{(1.2\,\frac{m}{s})^2}\right] \times 100\% = 26.2\%$$

Example 7: A bullet in a block on a horizontal surface

A $10g$ projectile is fired at $500\frac{m}{s}$ into a $1kg$ block sitting on a frictionless horizontal surface. The projectile lodges in the center of the block, and both move off together.

a. What is the final velocity of the block after the collision?

b. The block slides along the frictionless surface some distance and then encounters a ramp, which slopes up at an angle of 60^0. What distance does the block travel along the surface of the ramp before coming to a stop?

c. If the coefficient of friction between the block and the ramp is $\mu = 0.2$, how far does the block slide up the ramp before stopping?

Solution:

a. Assuming that the positive x-direction is to the right we apply conservation of momentum. We find for the velocity after the collision

$$p_{ix} = p_{fx} \rightarrow m_b v_i = (m_b + m_{bl})V \rightarrow V = \frac{m_b v_i}{(m_b + m_{bl})} = \frac{0.01kg \times 500\frac{m}{s}}{1.01kg} = 4.95\frac{m}{s}$$

to the right.

b. Define d as the distance the block slides along the ramp and h as the height the block rises above the horizontal, we have from the geometry $\sin\theta = \dfrac{h}{d} \rightarrow h = d\sin\theta$. Applying conservation of energy between the bottom of the ramp and where the block comes to rest we have

$$\Delta E = 0 = \Delta U_g + \Delta KE = \left(mgy_f - mgy_i\right) + \left(\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2\right)$$

$$0 = mgh - \tfrac{1}{2}mv_i^2 = mgd\sin\theta - \tfrac{1}{2}mv_i^2$$

$$\therefore d = \sqrt{\frac{v_i^2}{2g\sin\theta}} = \sqrt{\frac{\left(4.95\frac{m}{s}\right)^2}{2\times 9.8\frac{m}{s^2}\sin 60}} = 1.44m$$

c. In the presence of friction, energy is lost to heat between the surfaces. To calculate the new distance we use

$$\Delta U_g + \Delta KE = -\Delta E_{friction}$$

$$\left(mgy_f - mgy_i\right) + \left(\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2\right) = -F_{friction} \times d_{new}$$

$$mgh - \tfrac{1}{2}mv_i^2 = mgd_{new}\sin\theta - \tfrac{1}{2}mv_i^2 = -\mu_k mg\cos\theta d_{new}$$

$$\therefore d_{new} = \sqrt{\frac{v_i^2}{2g(\sin\theta + \mu_k\cos\theta)}} = \sqrt{\frac{\left(4.95\frac{m}{s}\right)^2}{2\times 9.8\frac{m}{s^2}(\sin 60 + 0.2\cos 60)}} = 1.29m$$

Example 8: Fireworks

A rocket used for fireworks explodes just when it reaches its highest point in a vertical trajectory. It initially bursts into three fragments with masses of m, $3m$, and $4m$, each of these to explode slightly later. If the $4m$ fragment falls vertically downward with an initial velocity of $8\frac{m}{s}$, and the $3m$ fragment is ejected with a velocity of $10\frac{m}{s}$ at an angle of 30^0 above the horizontal, what is the velocity of the third fragment?

Solution: At the highest point of the rocket's motion, its velocity is zero. Therefore the initial x- and initial y-momenta are both zero when the rocket explodes. After the explosion we apply conservation of momentum in the vertical and horizontal directions. Assuming that the piece of mass m has a momentum in the same quadrant as the $3m$ piece.

In the vertical direction $0 = -4m\left(8\frac{m}{s}\right) + 3m\left(10\frac{m}{s}\right)\sin 30 + mv\sin\phi \rightarrow 17 = v\sin\phi$.

In the horizontal direction we have
$0 = 3m\left(10\frac{m}{s}\right)\cos 30 + mv\cos\phi \rightarrow -25.98 = v\sin\phi$.

Here we have that the x- component of the velocity is negative, while the y-component is positive, so the momentum vector lies in the 2$^{\text{nd}}$ quadrant, our guess was incorrect, but that's ok. Now we know. Taking the ratio of these two equations produces an angle of 33.2^0 above the negative x-axis. Then using any one of the above equations we find for the magnitude of the velocity to be $31\frac{m}{s}$.

Example 9: Proton scattering

A proton moving with an initial velocity v_{ix} in the x-direction, as shown in Figure 11, collides elastically with another proton that is initially at rest. If the two protons have equal speeds after the collision, what is the speed of each proton after the collision in terms of v_{ix}, and what are the directions of the velocity vector after the collision?

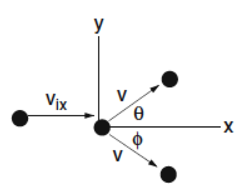

Figure 11: An incident proton scattering off a stationary proton.

Solution: We break up the momentum into x and y-components and use conservation of momentum in each direction. We have in the x-direction $p_{ix} = p_{fx} \rightarrow mv_{ix} = mv\cos\theta + mv\cos\phi$, while in the y-direction $p_{iy} = p_{fy} \rightarrow 0 = mv\sin\theta - mv\sin\phi \rightarrow \sin\theta = \sin\phi \rightarrow \theta = \phi$.

Using the results from the vertical motion we rewrite the x-momentum as $v_{ix} = v\cos\theta + v\cos\phi = 2v\cos\phi$.

Next we use the kinetic energy to obtain an expression for v_{ix} in terms of v so that we can determine the unknown angle ϕ. Conservation of energy gives $KE_i = KE_f \rightarrow \frac{1}{2}mv_{ix}^2 = \frac{1}{2}mv^2 + \frac{1}{2}mv^2 \rightarrow v_{ix}^2 = 2v^2 \rightarrow v_{ix} = \sqrt{2}v$.

Therefore we have the magnitude of the velocity after the collision as $v = \dfrac{v_{ix}}{\sqrt{2}}$. And from the x-momentum we calculate the angle to be $v_{ix} = \sqrt{2}v = 2v\cos\phi \rightarrow \cos\phi = \dfrac{\sqrt{2}}{2} \rightarrow \phi = 45^o$. The angle of the velocity vector after the collision is $\phi = \theta = 45^0$.