Physics 220

Exam #1

April 22, 2016

Name____________________________________

Please read and follow these instructions carefully:

- Read all problems carefully before attempting to solve them.
- Your work must be legible, and the organization clear.
- You must show all work, including correct vector notation.
- You will not receive full credit for correct answers without adequate explanations.
- You will not receive full credit if incorrect work or explanations are mixed in with correct work. So erase or cross out anything you don't want graded.
- Make explanations complete but brief. Do not write a lot of prose.
- Include diagrams.
- Show what goes into a calculation, not just the final number. For example $|\vec{p}| \approx m|\vec{v}| = (5kg)\times(2\,\frac{m}{s}) = 10\,\frac{kg\cdot m}{s}$
- Give standard SI units with your results unless specifically asked for a certain unit.
- Unless specifically asked to derive a result, you may start with the formulas given on the formula sheet including equations corresponding to the fundamental concepts.
- Go for partial credit. If you cannot do some portion of a problem, invent a symbol and/or value for the quantity you can't calculate (explain that you are doing this), and use it to do the rest of the problem.
- Each free-response part is worth 10 points

Problem #1	/30
Problem #2	/30
Problem #3	/30
Problem #4	/30
Total	/120

I affirm that I have carried out my academic endeavors with full academic honesty.

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1. An electron in an infinite potential well makes a transition from the $n = 3$ state to the ground state. In doing so, a photon is emitted with wavelength $\lambda = 20.9nm$.

   a. What is the width of the potential well?

$$\Delta E = E_u - E_l = \left(n_u^2 - n_l^2\right)\frac{\pi^2\hbar^2}{2ma^2} = \frac{2\pi\hbar c}{\lambda}$$
$$a = \sqrt{\frac{\pi\hbar\lambda}{4mc}\left(n_u^2 - n_l^2\right)} = \sqrt{\frac{6.63\times10^{-34}\,Js\times20.9\times10^{-9}\,m}{8\times9.11\times10^{-31}\,kg\times3\times10^{8}\,\frac{m}{s}}\left(3^2-1^2\right)} = 2.25\times10^{-10}\,m$$

   b. Suppose that many of these emitted photons were incident on a sodium surface $\phi_{Na} = 2.3eV$. What stopping potential would be needed to stop the most energetic electrons from reaching the collector?

$$E = \frac{hc}{\lambda} = \frac{6.63\times10^{-34}\,Js\times3\times10^{8}\,\frac{m}{s}}{20.9\times10^{-9}\,m}\times\frac{1eV}{1.6\times10^{-19}\,J} = 59.5eV$$

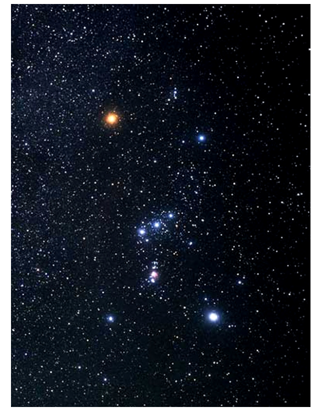
$$eV_{stop} = E - \phi = 59.5eV - 2.28eV = 57.2eV$$
$$\therefore V_{stop} = 57.2V$$

   c. Suppose that you could construct a laser beam from this situation and that the laser beam was directed onto the sodium surface and made a spot of diameter $d = 1mm$. What photocurrent would be produced by this laser light source if the laser were operated at $5mW = 5\times10^{-3}W$? Hint: The photocurrent is given as $\frac{\Delta Q}{\Delta t}$ and assume that for every photon incident, one electron is liberated from the surface.

$$S = \frac{P}{A} = \frac{E}{tA} = \frac{N_{photons}\left(\frac{hc}{\lambda}\right)}{tA} \rightarrow$$
$$\frac{N_{photons}}{t} = \frac{P}{\left(\frac{hc}{\lambda}\right)} = \frac{5\times10^{-3}W}{\dfrac{6.63\times10^{-34}\,Js\times3\times10^{8}\,\frac{m}{s}}{20.9\times10^{-9}\,m}} = 5.25\times10^{14}$$

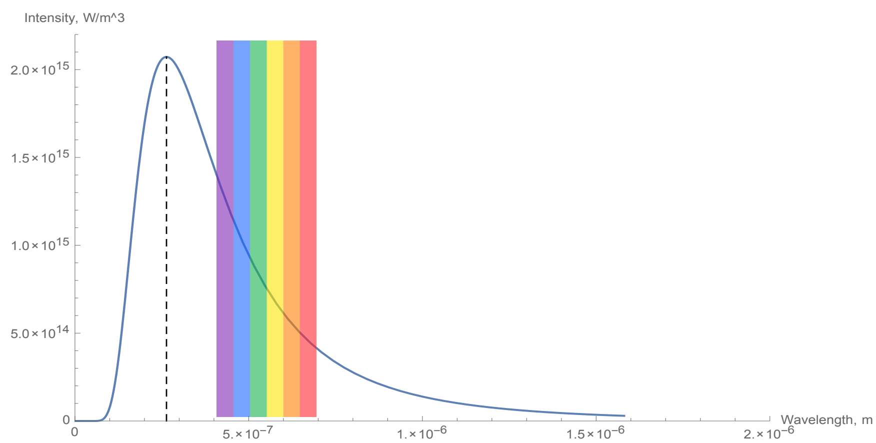
$$\frac{\#_{photons}}{s} = \frac{\#_{e^-}}{s} = 5.25\times10^{14}$$
$$I = \frac{\Delta Q}{\Delta t} = 5.25\times10^{14}\,\frac{e^-}{s}\times\frac{1.6\times10^{-19}\,C}{1e^{-1}} = 8.4\times10^{-5}\,A = 84\mu A$$

2. The Constellation Orion is one of the brightest and best-known constellations in the night sky. Orion has been known since ancient times and the constellation is also known as the Hunter, as it is associated with one in Greek mythology. As seen in the figure on the right Betelgeuse is the red giant in the upper left corner and the blue giant Rigel is in the lower right. The constellation Orion contains two of the ten brightest stars in the sky – Rigel (Beta Orionis) and Betelgeuse (Alpha Orionis) – a number of famous nebulae – the Orion Nebula (Messier 42), De Mairan's Nebula (Messier 43) and the Horsehead Nebula, among others – the well-known Trapezium Cluster, and one of the most prominent asterisms in the night sky – Orion's Belt.



http://www.constellation-guide.com/constellation-list/orion-constellation/

a. Given the blackbody spectrum of the star Rigel in the constellation Orion, what is the surface temperature of the star?



From the graph, $\lambda_{\max} = 2.6\times10^{-7}\,m$.

$$T = \frac{2.9\times10^{-3}\,mK}{\lambda_{\max}} = \frac{2.9\times10^{-3}\,mK}{2.6\times10^{-7}\,m} = 11154K$$

b. Given the information in the table below, what is the luminosity of the star. Hint: the luminosity is the power output of a star.

Mass	$3.4\times10^{31}\,kg$
Radius	$5.4\times10^{8}\,m$
Distance to Earth	$7.3\times10^{19}\,m$
Age	$8.0\times10^{6}\,yrs$
Apparent Magnitude	0.13
Rotational Velocity	$25\,\frac{km}{s}$

$$S=\sigma T^4=\frac{P}{A}\rightarrow P=A\sigma T^4$$

$$P=4\pi\left(5.4\times10^{8}\,m\right)^2\times5.67\times10^{-8}\,\tfrac{W}{m^2K^4}\times\left(11154K\right)^4$$

$$P=3.22\times10^{27}\,W$$

c. Betelgeuse is a red super giant star and it is the star in the upper left corner of the figure. If Betelgeuse has a surface temperature of $3500K$, by what factor does the energy output per unit time per unit area of Rigel exceed that of Betelgeuse?

$$\frac{S_R}{S_B}=\frac{\sigma T_R^4}{\sigma T_B^4}=\left(\frac{T_R}{T_B}\right)^4=\left(\frac{11154K}{3500K}\right)^4=103$$

3. Suppose that you have a particle of mass m in a state given by the wave function $\Psi(x,t) = Ae^{-a\left[\frac{mx^2}{\hbar}+it\right]}$ where A and a are positive real constants.

   a. Determine the normalization constant A.

$$1 = \int_{-\infty}^{\infty} \Psi^* \Psi \, dx = \int_{-\infty}^{\infty} \left(Ae^{-a\left[\frac{mx^2}{\hbar}-it\right]} \right)\left(Ae^{-a\left[\frac{mx^2}{\hbar}+it\right]} \right) dx$$

$$1 = \int_{-\infty}^{\infty} A^2 e^{-\frac{2amx^2}{\hbar}} \, dx = A^2 \left(\sqrt{\frac{\pi\hbar}{2ma}} \right)$$

$$A = \left(\frac{2ma}{\pi\hbar} \right)^{\frac{1}{4}}$$

   b. What potential energy function would this particle of mass m have to move in order to have $\Psi(x,t) = Ae^{-a\left[\frac{mx^2}{\hbar}+it\right]}$ as a wave function?

$$\Psi(x,t) = Ae^{-a\left[\frac{mx^2}{\hbar}+it\right]} = Ae^{\frac{-amx^2}{\hbar}} e^{-ait}$$

$$\rightarrow \frac{d\Psi}{dt} = -iaAe^{\frac{-amx^2}{\hbar}} e^{-ait} = -ia\Psi$$

$$\frac{d\Psi}{dx} = -\frac{2ma}{\hbar} xAe^{\frac{-amx^2}{\hbar}} e^{-ait} = -\frac{2ma}{\hbar} x\Psi$$

$$\frac{d^2\Psi}{dx^2} = \frac{d}{dx}\left(\frac{d\Psi}{dx} \right) = -\frac{2ma}{\hbar}\Psi + \left(-\frac{2ma}{\hbar} x \right)\left(-\frac{2ma}{\hbar} x \right) Ae^{\frac{-amx^2}{\hbar}} e^{-ait}$$

$$\rightarrow \frac{d^2\Psi}{dx^2} = -\frac{2ma}{\hbar}\Psi + \frac{4m^2a^2}{\hbar^2} x^2\Psi$$

$SWE:$

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + V\Psi = i\hbar\frac{d\Psi}{dt}$$

$$-\frac{\hbar^2}{2m}\left[-\frac{2ma}{\hbar} + \frac{4m^2a^2}{\hbar^2} x^2\Psi \right]\Psi + V\Psi = i\hbar[-ia]\Psi$$

$$\rightarrow \left[a\hbar - 2ma^2x^2 \right] + V = a\hbar$$

$$\therefore V = 2ma^2x^2$$

c. What are the expectation values of $\langle p^2 \rangle$ and $\langle E \rangle$?

$$\langle p^2 \rangle = \int \Psi^* \hat{p}^2 \Psi \, dx = -\hbar^2 \int \Psi^* \frac{d^2\Psi}{dx^2} dx$$

$$\langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \left[-\frac{2ma}{\hbar}\Psi + \frac{4m^2a^2}{\hbar^2} x^2 \Psi \right] dx$$

$$\langle p^2 \rangle = 2ma\hbar \int_{-\infty}^{\infty} \Psi^*\Psi dx - 4m^2a^2 \int_{-\infty}^{\infty} x^2 \Psi^* \Psi dx$$

$$\langle p^2 \rangle = 2ma\hbar - 4m^2a^2 \left(\sqrt{\frac{2ma}{\pi\hbar}} \right) \left[\frac{\hbar}{4ma} \sqrt{\frac{\pi\hbar}{2ma}} \right] = 2ma\hbar - ma\hbar$$

$$\therefore \langle p^2 \rangle = ma\hbar$$

The expectation value of the energy can be done in two ways.

Method 1:

$$\langle E \rangle = \int \Psi^* \hat{E} \Psi \, dx = i\hbar \int \Psi^* \frac{d\Psi}{dt} dx$$

$$\langle E \rangle = i\hbar(-ia) \int_{-\infty}^{\infty} \Psi^* \Psi dx = a\hbar$$

Method 2:

$$\langle E \rangle = \langle H \rangle = \frac{\langle p^2 \rangle}{2m} + \langle V \rangle$$

$$\langle V \rangle = \int \Psi^* \hat{V} \Psi \, dx = 2ma^2 \int \Psi^* x^2 \Psi \, dx = 2ma^2 \left[\left(\sqrt{\frac{2ma}{\pi\hbar}} \right) \left[\frac{\hbar}{4ma} \sqrt{\frac{\pi\hbar}{2ma}} \right] \right] = \frac{a\hbar}{2}$$

$$\langle E \rangle = \langle H \rangle = \frac{\langle p^2 \rangle}{2m} + \langle V \rangle = \frac{ma\hbar}{2m} + \frac{a\hbar}{2} = a\hbar$$

4. When we examined the infinite well in class, the well was positioned so that the left edge of the well was at $x=0$ and the right edge was at $x=a$ and when we did the finite well we put the left and right edges at $x=-a$ and $x=a$ respectively. Of course you can put the well anywhere and make it any size you'd like. Consider the "centered" infinite well with its left edge at $x=-a$ and its right edge at $x=a$. The potential function is given as shown below.

$$V(x)=\begin{cases} \infty \ for \ x<-a \\ 0 \ for \ -a\le x\le a \\ \infty \ for \ x>a \end{cases}$$

a. What are the normalized wave functions to the time independent Schrodinger wave equation in each region?

For $x<-a$ and $x>+a$ the wave function does not exist, therefore $\psi_1=\psi_3=0$. The only place we have a non-zero wave function is in the well.

In the well:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2}=E\psi_2 \rightarrow \frac{d^2\psi_2}{dx^2}=-\frac{2mE}{\hbar^2}\psi_2=-k^2\psi_2; \quad k=\frac{\sqrt{2mE}}{\hbar}$$

$$\therefore \psi_2=A\sin(kx)+B\cos(kx)$$

$$\psi(x)=\begin{cases} 0 & x<-a \\ A\sin(kx)+B\cos(kx) & -a\le x\le +a \\ 0 & x>+a \end{cases}$$

Apply boundary conditions: $\psi(x=-a)=\psi(x=+a)=0$

At $x<-a$: $-A\sin(ka)+B\cos(ka)=0$

At $x>+a$: $A\sin(ka)+B\cos(ka)=0$

Both cannot be true at the same time, so let's add and subtract the solutions.

Adding and assuming that $A = 0$ and $B \neq 0$:
$2B\cos(ka) = 0 \rightarrow B\cos(ka) = 0 \rightarrow ka = (n - \tfrac{1}{2})\pi; \;\; n = 1,2,3...$

$$k = (n - \tfrac{1}{2})\frac{\pi}{a} = (2n-1)\frac{\pi}{2a}$$

Let $j = (2n-1)$ where $j = 1,3,5... \rightarrow k_j = \dfrac{j\pi}{2a}; \;\; j = odd$

$$\therefore \psi_2 = B\cos\left(k_j x\right); \;\; j = odd$$

$$1 = \int_{-a}^{a} B^2 \cos^2\left(\frac{j\pi}{2a}x\right)dx = B^2\left(a + \frac{a\sin\left(j\pi\right)}{j\pi}\right) = aB^2 \rightarrow B = \sqrt{\frac{1}{a}}$$

$$\psi_2 = \sqrt{\frac{1}{a}}\cos\left(k_j x\right); \;\; j = odd$$

Subtracting and assuming that $A \neq 0$ and $B = 0$:
$2A\sin(ka) = 0 \rightarrow A\cos(ka) = 0 \rightarrow ka = n\pi; \;\; n = 1,2,3...$

$$k = \left(2n\right)\frac{\pi}{2a}$$

Let $j = 2n$ where $j = 2,4,6,.. \rightarrow k_j = \dfrac{j\pi}{2a}; \;\; j = even$

$$\therefore \psi_2 = A\sin\left(k_j x\right); \;\; j = even$$

$$1 = \int_{-a}^{a} A^2 \sin^2\left(\frac{j\pi}{2a}x\right)dx = A^2\left(a - \frac{a\sin\left(j\pi\right)}{j\pi}\right) = aA^2 \rightarrow A = \sqrt{\frac{1}{a}}$$

$$\psi_2 = \sqrt{\frac{1}{a}}\sin\left(k_j x\right); \;\; j = even$$

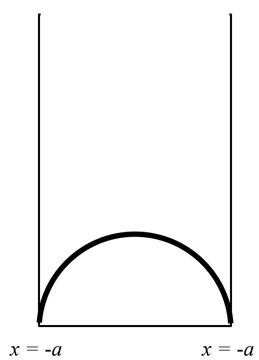
b. What is the energy of any state?

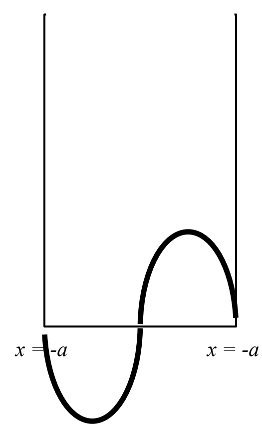
$$k = \frac{j\pi}{2a} = \frac{\sqrt{2mE}}{\hbar}$$

$$E_j = j^2\left(\frac{\pi^2\hbar^2}{8ma^2}\right)$$

c. Sketch (with appropriate dimensions) the ground and first excited state wave functions.

$$\psi_{21} = \sqrt{\frac{1}{a}}\cos\left(\frac{\pi}{2a}x\right) \qquad \psi_{22} = \sqrt{\frac{1}{a}}\sin\left(\frac{2\pi}{2a}x\right)$$





Physics 220 Equations

Useful Integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int_{-a/2}^{a/2} \cos^2(qx)dx = \frac{a}{2} + \frac{\sin(aq)}{2q}$$

$$\int_{-a/2}^{a/2} \sin^2(qx)dx = \frac{a}{2} - \frac{\sin(aq)}{2q}$$

$$\int_{-a/2}^{a/2} x\cos^2(qx)dx = 0$$

$$\int_{-a/2}^{a/2} x\sin^2(qx)dx = 0$$

$$\int_{-a/2}^{a/2} \sin(qx)\cos(qx)dx = 0$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a}$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{\infty} xe^{-ax^2}\,dx = 0$$

$$\int_{-\infty}^{\infty} x^2e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{2a^{\frac{3}{2}}}$$

$$\int_{400nm}^{700nm} \frac{2\pi hc^2}{\lambda^5}\left[\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right]d\lambda = 1197\,\tfrac{W}{m^2}$$

Constants:

$g = 9.8\,\frac{m}{s^2}$

$G = 6.67\times10^{-11}\,\frac{Nm^2}{kg^2}$

$c = 3\times10^{8}\,\frac{m}{s}$

$\sigma = 5.67\times10^{-8}$

$k_B = 1.38\times10^{-23}\,\frac{J}{K}$

$1eV = 1.6\times10^{-19}\,J$

$1e = 1.6\times10^{-19}\,C$

$h = 6.63\times10^{-34}\,Js;$

$m_e = 9.11\times10^{-31}\,kg = 0.511\,\frac{MeV}{c^2}$

$m_p = 1.67\times10^{-27}\,kg = 938\,\frac{MeV}{c^2}$

$m_n = 1.69\times10^{-27}\,kg = 939\,\frac{MeV}{c^2}$

$m_E = 6\times10^{24}\,kg$

$R_E = 6.4\times10^{6}\,m$

Formulas:

$$c = \upsilon\lambda$$

$$E = h\upsilon = \frac{hc}{\lambda}$$

$$\frac{dS}{d\lambda} = \frac{2\pi hc^2}{\lambda^5}\left[\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right]$$

$$\frac{dS}{d\upsilon} = \frac{2\pi h\upsilon^3}{c^2}\left(\frac{1}{e^{\frac{h\upsilon}{kT}}-1}\right)$$

$$\frac{dS}{d\lambda} = \frac{2\pi ckT}{\lambda^4}$$

$$\lambda_{\max} = \frac{2.9\times10^{-3}\,m\cdot K}{T}$$

$$S = \sigma T^4$$

$$eV_{stop} = hf - \phi$$

$$\lambda' = \lambda + \frac{h}{mc}(1-\cos\phi)$$

$$\hbar = \frac{h}{2\pi};\; k = \frac{2\pi}{\lambda};\; \omega = 2\pi f$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t} = E\psi$$

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

$$\hat{p} = -i\hbar\frac{\partial}{dx}$$

$$\hat{T} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V$$

$$\hat{x} = x$$

$$\langle O\rangle = \int \psi^*\hat{O}\psi\,dr$$

$$P = \int \psi^*\psi\,dr$$

$$E_n = n^2\left(\frac{\pi^2\hbar^2}{2ma^2}\right)$$

$$\Psi_n(x,t) = \sqrt{\frac{2}{a}}\sin(k_n x)e^{-i\frac{E_n}{\hbar}t}$$