Physics 220

Exam #1

April 21, 2017

Name___________________________________

Please read and follow these instructions carefully:

- Read all problems carefully before attempting to solve them.
- Your work must be legible, and the organization clear.
- You must show all work, including correct vector notation.
- You will not receive full credit for correct answers without adequate explanations.
- You will not receive full credit if incorrect work or explanations are mixed in with correct work. So erase or cross out anything you don't want graded.
- Make explanations complete but brief. Do not write a lot of prose.
- Include diagrams.
- Show what goes into a calculation, not just the final number. For example $|\vec{p}| \approx m|\vec{v}| = (5kg)\times(2\,\tfrac{m}{s}) = 10\,\tfrac{kg\cdot m}{s}$
- Give standard SI units with your results unless specifically asked for a certain unit.
- Unless specifically asked to derive a result, you may start with the formulas given on the formula sheet including equations corresponding to the fundamental concepts.
- Go for partial credit. If you cannot do some portion of a problem, invent a symbol and/or value for the quantity you can't calculate (explain that you are doing this), and use it to do the rest of the problem.
- Each free-response part is worth 10 points

Problem #1	/40
Problem #2	/40
Total	/80

I affirm that I have carried out my academic endeavors with full academic honesty.

____

1. Consider the system show below that is described by the potential fucntion

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \le x \le a \\ V_0 & x > a \end{cases}$$

   a. What are the allowed solutions to the time independent Schrödinger wave equation inside the well and inside of the barrier? Express your answers to the different regions in terms of the overall constant from the solutions inside of the well.

      In the region $x < 0$, there is no wave function.
      In the region $0 \le x \le a$, we have
      $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \rightarrow \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi$ with solutions
      $\psi_1 = A\sin kx + B\cos kx$.
      In the region $x > a$, we have
      $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi \rightarrow \frac{d^2\psi}{dx^2} = -\frac{2m(E - V_0)}{\hbar^2}\psi = \frac{2m(V_0 - E)}{\hbar^2}\psi = (k')^2\psi$ with solutions $\psi_2 = Ce^{k'x} + De^{-k'x}$.

      To keep finite ψ_2, we require that as $x \rightarrow \infty$, $\psi_2 \rightarrow 0$, so $C = 0$. Therefore $\psi_2 = De^{-k'x}$.

      Thus the wave functions are:

$$\psi(x) = \begin{cases} A\sin kx + B\cos kx & 0 \le x \le a \\ De^{-k'x} & x > a \end{cases}$$

      Applying the boundary conditions that the wave function and its first derivative are continuous at $x = a$ and the wave function vanishes at $x = 0$ we have:

$$\psi\Big|_{x=0} : A\sin 0 + B\cos 0 = 0 \rightarrow B = 0$$

$$\psi\Big|_{x=a} : A\sin ka = De^{-k'a} \rightarrow D = Ae^{k'a}\sin ka$$

$$\frac{d\psi}{dx}\Big|_{x=a} : Ak\cos ka = -k'De^{-k'a}$$

      Thus the wave functions are:

$$\psi(x) = \begin{cases} A\sin kx & 0 \le x \le a \\ Ae^{k'a}\sin(ka)e^{-k'x} & x > a \end{cases}$$

b. What is the normalization constant in terms of α and β, where we define $\alpha = ka$ and $\beta = k'a$.

To normalize we apply the normalization condition:

$$P = \int \psi^* \psi \, dx = \int_0^a A^2 \sin^2\left(\alpha \frac{x}{a}\right) dx + \int_a^\infty A^2 \sin^2(\alpha) e^{2\beta} e^{-2\beta\frac{x}{a}} \, dx = 1$$

$$1 = A^2 \left(\frac{x}{2} - \frac{a \sin\left(\frac{2\alpha x}{a}\right)}{4\alpha} \right)\Bigg|_0^a + A^2 \sin^2(\alpha) e^{2\beta} \frac{a}{2\beta} e^{-2\beta\frac{x}{a}} \Bigg|_a^\infty$$

$$1 = A^2 \left(\frac{a}{2} - \frac{a}{4\alpha} \sin 2\alpha + \frac{a}{2\beta} \sin^2 \alpha \right)$$

$$A = \sqrt{\frac{2}{a}} \left(1 - \frac{\sin 2\alpha}{2\alpha} + \frac{\sin^2 \alpha}{\beta} \right)^{-\frac{1}{2}}$$

This could also have been done on Mathematica. The code is below and if you use the definitions of α and β provided the result is identical to A above.

```
Integrate[A^2*Sin[k*a]^2*Exp[2*kprime*a]*Exp[-2*kprime*x], {x, a, Infinity}]

ConditionalExpression[A^2 Sin[a k]^2 / (2 kprime), Re[kprime] > 0]


rightwell = A^2 Sin[a k]^2 / (2 kprime)
Solve[(leftwell + rightwell) == 1, A]
A^2 Sin[a k]^2 / (2 kprime)

{{A -> -2/Sqrt[2 a + (2 Sin[a k]^2)/kprime - Sin[2 a k]/k]}, {A -> 2/Sqrt[2 a + (2 Sin[a k]^2)/kprime - Sin[2 a k]/k]}}
```

c. In terms of V_0, what is the energy of the single bound state for $r = 2$? Hint define $\alpha = ka$ and $\beta = k'a$. If you cannot determine a value for alpha, use $\alpha = 1$.

From the boundary conditions at $x = a$ we get:

$\psi\big|_{x=a}$: $A\sin ka = De^{-k'a}$

$\left.\dfrac{d\psi}{dx}\right|_{x=a}$: $Ak\cos ka = -k'De^{-k'a}$

Dividing these two expressions produces the equation for determining the bound state energies.

$$\frac{A\sin ka}{Ak\cos ka} = \frac{De^{-k'a}}{-k'De^{-k'a}} \rightarrow -k' = k\cot ka \rightarrow -k'a = ka\cot ka \Rightarrow -\beta = \alpha\cot\alpha .$$

Following the procedure from class, we define $\alpha^2 + \beta^2 = r^2$, so that

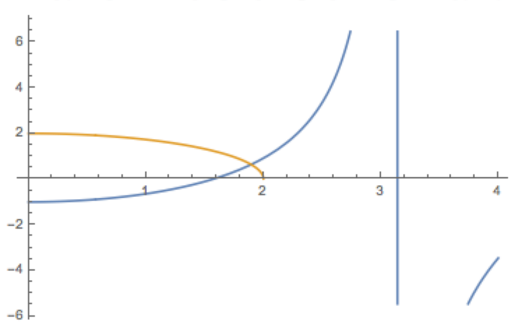
$$\left(\frac{\sqrt{2mE}a}{\hbar}\right)^2 + \left(\frac{\sqrt{2m(V_0 - E)}a}{\hbar}\right)^2 = \frac{2mV_0a^2}{\hbar^2} = r^2$$, and thus $\beta = \sqrt{r^2 - \alpha^2}$. To solve this transcendental equation for alpha we plot the two functions and look for intersections. From the graph below (generated on Mathematica) and the FindRoot command, $\alpha = 1.8955$.

From alpha we can determine the energy of the single odd bound state.

$$\alpha^2 = \left(\frac{\sqrt{2mE}a}{\hbar}\right)^2 = \left(\frac{r^2}{V_0}\right)E$$

$$\rightarrow E = \left(\frac{\alpha}{r}\right)^2 V_0 = \left(\frac{1.8955}{2}\right)^2 V_0 = 0.898V_0$$

```
Clear[alpha, beta, r]
Plot[{-alpha*Cot[alpha], Sqrt[4 - alpha^2]}, {alpha, 0, 4}]
```




```
FindRoot[-alpha*Cot[alpha] == Sqrt[4 - alpha^2], {alpha, 1.9}]
{alpha → 1.89549}
```

d. What is the expectation value of the position? What does the result tell you?

$$\langle x \rangle = \int \psi^* x \psi \, dx = \int_0^a A^2 x \sin^2\left(\alpha \frac{x}{a}\right) dx + \int_a^\infty A^2 \sin^2(\alpha) e^{2\beta} x e^{-2\beta \frac{x}{a}} dx$$

Evaluating the normalization coefficient:

$$A = \sqrt{\frac{2}{a}} \left(1 - \frac{\sin(2\times 1.8955)}{(2\times 1.8955)} + \frac{\sin^2(1.8955)}{\sqrt{4-(1.8955)^2}}\right)^{-\frac{1}{2}} = 0.624\sqrt{\frac{2}{a}}$$

And,

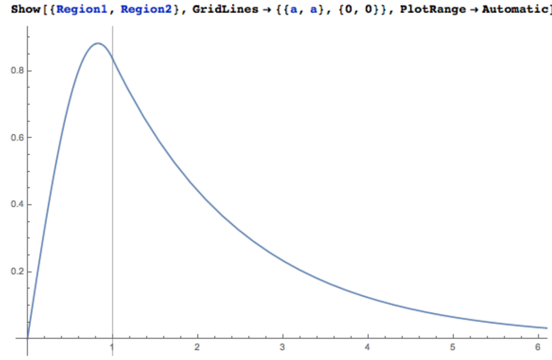
$$\langle x \rangle = \int \psi^* x \psi \, dx = \int_0^a A^2 x \sin^2\left(\alpha \frac{x}{a}\right) dx + \int_a^\infty A^2 \sin^2(\alpha) e^{2\beta} x e^{-2\beta \frac{x}{a}} dx = 1.28a$$

I chose to use Mathematica to evaluate the numbers in the problem and the integrals. The code and solution are below. What this result implies is that if you made a huge ensemble of identical particles and measured their positions you'd find on average that the particle is most likely to be found outside of the well rather than inside, even though its energy is less than the barrier height.

```
In[40]:= Clear[alpha, r, beta, x, a, left, right, expx, A]
         r := 2
         alpha = 1.8955
         beta = Sqrt[r^2 - alpha^2]
         A = (Sqrt[2 / a]) * (1 - (Sin[2 * alpha] / (2 * alpha)) + (Sin[alpha] ^ 2 / beta)) ^ (-1 / 2)
         right = Integrate[x * (A * Sin[alpha * x / a]) ^ 2, {x, 0, a}]
         left = Integrate[x * (A * Sin[alpha] * Exp[beta] * Exp[-beta * x / a]) ^ 2, {x, a, Infinity}]
         expx = left + right

Out[42]= 1.8955

Out[43]= 0.638028

Out[44]= 0.882622 Sqrt[1/a]

Out[45]= 0. + 0.305576 a

Out[46]= ConditionalExpression[0.978085 a, Re[a] > 0]

Out[47]= ConditionalExpression[0. + 1.28366 a, Re[a] > 0]
```

In case you're interested, the graph of the wave function is below.

```
Clear[alpha, r, beta, x, a]
a := 1
r := 2
alpha := 1.8955
beta := Sqrt[r^2 - alpha^2]
A := (Sqrt[2 / a]) * (1 - (Sin[2 * alpha] / (2 * alpha)) + (Sin[alpha] ^ 2 / beta)) ^ (-1 / 2)
Region1 := Plot[A * Sin[alpha * x / a], {x, 0, a}]
Region2 := Plot[A * Sin[alpha] * Exp[beta] * Exp[-beta * x / a], {x, a, 10 * a}]
Show[{Region1, Region2}, GridLines → {{a, a}, {0, 0}}, PlotRange → Automatic]
```



2. Suppose that the state of a particle of mass m is given by the normalized wave function $|\Psi(x,t)\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \sqrt{\frac{2m\omega}{\hbar}} x e^{-\frac{m\omega}{2\hbar}x^2} e^{-i\frac{E}{\hbar}t}$.

   a. Determine the expression for the expectation value of the kinetic energy, $\langle T \rangle$.

      Hint: To make your calculations easier in parts a, b and c, define $\alpha = \frac{m\omega}{\hbar}$.

$$|\Psi(x,t)\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \sqrt{\frac{2m\omega}{\hbar}} x e^{-\frac{m\omega}{2\hbar}x^2} e^{-i\frac{E}{\hbar}t} = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \sqrt{2\alpha}\, x e^{-\frac{\alpha}{2}x^2} e^{-i\frac{E}{\hbar}t}$$

$$\langle T \rangle = \frac{\langle p^2 \rangle}{2m} = -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \Psi^*(x,t) \left[\frac{d^2\Psi(x,t)}{dx^2}\right] dx = \frac{3\alpha\hbar^2}{4m} = \frac{3\hbar^2}{4m}\left(\frac{m\omega}{\hbar}\right) = \frac{3}{4}\hbar\omega$$

```
In[286]:= Clear[x, alpha, p, hbar, m]
          (* define the wave function in terms of alpha *)
          psi := ((alpha / Pi) ^ (1 / 4)) * Sqrt[2 * alpha] * x * Exp[-alpha * x ^ 2 / 2]
          (*  define the square momentum operator *)
          psquared := - (hbar ^ 2 / (2 * m)) * D[D[psi, x], x]
          (* evaluate the expectation value of the kinetic energy *)
          Integrate[psi * psquared, {x, -Infinity, Infinity}]
```

Out[289]= ConditionalExpression$\left[\frac{3\ \text{alpha hbar}^2}{4\ \text{m}}, \text{Re}[\text{alpha}] > 0\right]$

b. Determine the expression for the expectation value of the potential energy, $\langle V \rangle = \frac{m\omega^2}{2}\langle x^2 \rangle$.

$$\langle V \rangle = \frac{m\omega^2}{2}\langle x^2 \rangle = \frac{m\omega^2}{2}\int_{-\infty}^{\infty} \Psi^*(x,t)\left[x^2\Psi(x,t)\right]dx = \frac{3\alpha\hbar^2}{4m} = \frac{3\hbar^2}{4m}\left(\frac{m\omega}{\hbar}\right) = \frac{3}{4}\hbar\omega$$

```
In[294]:= Clear[x, alpha, p, hbar, m, w]
          (* define the wave function in terms of alpha *)
          psi := ((alpha / Pi) ^ (1 / 4)) * Sqrt[2 * alpha] * x * Exp[-alpha * x ^ 2 / 2]
          (*  define the square position operator *)
          xsquared := Integrate[(m * w ^ 2 / 2) * x ^ 2 * psi * psi, {x, -Infinity, Infinity}]
          (* evaluate the expectation value of the potential energy *)
          Integrate[psi * psquared, {x, -Infinity, Infinity}]

Out[297]= ConditionalExpression[3 alpha hbar^2 / (4 m), Re[alpha] > 0]
```

c. What is the expectation value for the energy of the state? How does your answer compare to the sum of $\langle T\rangle+\langle V\rangle$?

$$\langle H\rangle=\int\Psi^*\left(-\tfrac{\hbar^2}{2m}\tfrac{d^2}{dx^2}+\tfrac{m\omega^2}{2}x^2\right)\Psi dx=-\tfrac{\hbar^2}{2m}\int\Psi^*\tfrac{d^2\Psi}{dx^2}dx+\tfrac{m\omega^2}{2}\int\Psi^*x^2\Psi dx=\langle T\rangle+\langle V\rangle$$

$$\langle H\rangle=\langle T\rangle+\langle V\rangle=\frac{3}{4}\hbar\omega+\frac{3}{4}\hbar\omega=\frac{3}{2}\hbar\omega$$

d. What is the probability of finding the particle in the range $0\le x\le 1$? Let the $\alpha=5$.

$$P=\int_0^1\Psi^*\Psi\,dx=\int_0^1\left[\left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}}\sqrt{2\alpha}\,xe^{-\frac{\alpha}{2}x^2}e^{i\frac{E}{\hbar}t}\right]\left[\left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}}\sqrt{2\alpha}\,xe^{-\frac{\alpha}{2}x^2}e^{-i\frac{E}{\hbar}t}\right]dx$$

$$P=2\alpha\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}\int_0^1 x^2e^{-\alpha x^2}\,dx=-\sqrt{\frac{\alpha}{\pi}}\left(xe^{-\alpha x^2}\Big|_0^1\right)+\frac{\mathrm{Erf}\left[\sqrt{\alpha}x\right]}{2}\Bigg|_0^1$$

$$P=-\sqrt{\frac{\alpha}{\pi}}\left(e^{-\alpha}\right)+\left(\frac{\mathrm{Erf}\left[\sqrt{\alpha}\right]}{2}-\frac{\mathrm{Erf}[0]}{2}\right)=-\sqrt{\frac{5}{\pi}}\left(e^{-5}\right)+\left(\frac{\mathrm{Erf}\left[\sqrt{5}\right]}{2}-\frac{\mathrm{Erf}[0]}{2}\right)$$

$$P=-0.0085+\frac{1}{2}[0.9875-0]=0.485\sim 49\%$$

This can be evaluated on Mathematica. The code is below.

```
In[298]:= Clear[alpha, a, x, coef, exp, ans]
          (* defines the constant terms *)
          coef = Sqrt[alpha / Pi] * 2 * alpha
          (* defines and evaluates the integral *)
          exp = coef * Integrate[x ^ 2 * Exp[-alpha * x ^ 2], {x, 0, 1}]
```

Out[299]= $\frac{2\,\mathrm{alpha}^{3/2}}{\sqrt{\pi}}$

Out[300]= $\frac{2\,\mathrm{alpha}^{3/2}\left(-\frac{e^{-\mathrm{alpha}}}{2\,\mathrm{alpha}}+\frac{\sqrt{\pi}\,\mathrm{Erf}\left[\sqrt{\mathrm{alpha}}\right]}{4\,\mathrm{alpha}^{3/2}}\right)}{\sqrt{\pi}}$

```
          (* define alpha and numerically evaluate the error function *)
          alpha = 5
          N[exp]

Out[305]= 5

Out[306]= 0.490717
```

Physics 220 Equations

Useful Integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int \cos^2(qx)dx = \frac{x}{2} + \frac{\sin[2qx]}{4q}$$

$$\int \sin^2(qx)dx = \frac{x}{2} - \frac{\sin[2qx]}{4q}$$

$$\int \cos^3(qx)dx = \frac{3\sin[qx]}{4q} + \frac{\sin[3qx]}{12q}$$

$$\int \sin^3(qx)dx = -\frac{3\cos[qx]}{4q} + \frac{\cos[3qx]}{12q}$$

$$\int_{-a/2}^{a/2} x\cos^2(qx)dx = 0$$

$$\int_{-a/2}^{a/2} x\sin^2(qx)dx = 0$$

$$\int_{-a/2}^{a/2} \sin(qx)\cos(qx)dx = 0$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a}$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{\infty} xe^{-ax^2}\,dx = 0$$

$$\int_{-\infty}^{\infty} x^2e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{2a^{\frac{3}{2}}}$$

$$2\alpha\sqrt{\frac{\alpha}{\pi}}\int x^2e^{-\alpha x^2}\,dx = -\sqrt{\frac{\alpha}{\pi}}xe^{-\alpha x^2} + \frac{\text{Erf}\left[\sqrt{\alpha}x\right]}{2};\ \text{Erf}[0]\equiv 0$$

$$\int_{400nm}^{700nm} \frac{2\pi hc^2}{\lambda^5}\left[\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right]d\lambda = 1197\,\tfrac{W}{m^2}$$

Constants:

$g = 9.8\,\frac{m}{s^2}$

$G = 6.67\times10^{-11}\,\frac{Nm^2}{kg^2}$

$c = 3\times10^8\,\frac{m}{s}$

$\sigma = 5.67\times10^{-8}$

$k_B = 1.38\times10^{-23}\,\frac{J}{K}$

$1eV = 1.6\times10^{-19}\,J$

$1e = 1.6\times10^{-19}\,C$

$h = 6.63\times10^{-34}\,Js;$

$m_e = 9.11\times10^{-31}\,kg = 0.511\,\frac{MeV}{c^2}$

$m_p = 1.67\times10^{-27}\,kg = 938\,\frac{MeV}{c^2}$

$m_n = 1.69\times10^{-27}\,kg = 939\,\frac{MeV}{c^2}$

$m_E = 6\times10^{24}\,kg$

$R_E = 6.4\times10^6\,m$

$a = 0.5\times10^{-10}\,m$

Formulas:

$c = \upsilon\lambda$

$E = h\upsilon = \frac{hc}{\lambda}$

$$\frac{dS}{d\lambda} = \frac{2\pi hc^2}{\lambda^5}\left[\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right]$$

$$\frac{dS}{d\upsilon} = \frac{2\pi h\upsilon^3}{c^2}\left(\frac{1}{e^{\frac{h\upsilon}{kT}}-1}\right)$$

$$\frac{dS}{d\lambda} = \frac{2\pi ckT}{\lambda^4}$$

$$\lambda_{\max} = \frac{2.9\times10^{-3}\,m\cdot K}{T}$$

$S = \sigma T^4$

$eV_{stop} = hf - \phi$

$\lambda' = \lambda + \frac{h}{mc}(1-\cos\phi)$

$\hbar = \frac{h}{2\pi};\ k = \frac{2\pi}{\lambda};\ \omega = 2\pi f$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t} = E\psi$$

$\hat{E} = i\hbar\frac{\partial}{\partial t}$

$\hat{p} = -i\hbar\frac{\partial}{dx}$

$\hat{T} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$

$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V$

$\hat{x} = x$

$\langle O\rangle = \int \psi^*\hat{O}\psi\,dr$

$P = \int \psi^*\psi\,dx$

$E_n = n^2\left(\frac{\pi^2\hbar^2}{2ma^2}\right)$

$\Psi_n(x,t) = \sqrt{\frac{2}{a}}\sin(k_n x)e^{-i\frac{E_n}{\hbar}t}$

$$T = \frac{k'}{k}\left|\frac{F}{A}\right|^2$$

$$T + R = 1$$

$$T = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)}\sinh^2\left(\sqrt{\frac{8ma^2}{\hbar^2}(V_0 - E)}\right)};\;\; E < V_0$$

$$T = \frac{1}{1 + a^2k^2};\;\; E \sim V_0$$

$$T = \frac{1}{1 + \frac{V_0^2}{4E(E - V_0)}\sin^2\left(\sqrt{\frac{8ma^2}{\hbar^2}(E - V_0)}\right)};\;\; E > V_0$$

$$H_n(q) = (-1)^n e^{q^2}\left(\frac{d}{dq}\right)^n e^{-q^2};\quad q = \sqrt{\tfrac{m\omega}{2\hbar}}x$$

$$Y_l^{m_l}(\theta,\phi) = \varepsilon\sqrt{\frac{(2l+1)}{4\pi}\frac{(l - |m_l|)!}{(l + |m_l|)!}}e^{im\phi}P_l^{m_l}(\cos\theta);\quad \varepsilon = \begin{cases} (-1)^{m_l} & m_l \ge 0 \\ 1 & m_l \le 0 \end{cases}$$

$$P_l^{m_l}(\cos\theta) = \left(1 - \cos^2\theta\right)^{\frac{|m_l|}{2}}\left(\frac{d}{d\cos\theta}\right)^{|m_l|}P_l(\cos\theta)$$

$$P_l(\cos\theta) = \frac{1}{2^l l!}\left(\frac{d}{d\cos\theta}\right)^l\left(\cos^2\theta - 1\right)^l$$

$$L_{n-l-1}^{2l+1}\left(\tfrac{2r}{na}\right) = (-1)^{2l+1}\left(\tfrac{na}{2}\right)^{2l+1}\left(\tfrac{d}{dr}\right)^{2l+1}L_{n+2l}\left(\tfrac{2r}{na}\right)$$

$$L_{n+2l}\left(\tfrac{2r}{na}\right) = e^{\frac{2r}{na}}\left(\tfrac{na}{2}\right)^{n+2l}\left(\tfrac{d}{dr}\right)^{n+2l}\left(e^{-\frac{2r}{na}}\left(\tfrac{2r}{na}\right)^{n+2l}\right)$$

$$\left|\psi_{nlm_l}\right\rangle = \sqrt{\left(\tfrac{2}{na}\right)^3\frac{(n-l-a)!}{2n\left[(n+l)!\right]^3}}e^{-\frac{2r}{na}}\left(\tfrac{2r}{na}\right)^l\left[L_{n-l-1}^{2l+1}\left(\tfrac{2r}{na}\right)\right]Y_l^{m_l}(\theta,\phi)$$

$$a_\pm = \frac{1}{\sqrt{2m\hbar\omega}}\left(\mp ip + m\omega x\right)$$

$$H = \left(a_\pm a_\mp \pm \tfrac{1}{2}\right)\hbar\omega$$

$$L_\pm = L_x \pm iL_y$$

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$P = \int\psi^*\psi\, d^3r = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\psi^*\psi r^2\, dr\sin\theta\, d\theta d\phi$$