Physics 220

Exam #2

May 23 – May 30, 2014

Name___________________________________

Please read and follow these instructions carefully:

- Read all problems carefully before attempting to solve them.
- Your work must be legible, with clear organization, and you must show all work.
- You will not receive full credit if incorrect work or explanations are mixed in with correct work. So erase or cross out anything you don't want graded.
- Make explanations complete but brief. Do not write a lot of prose.
- Include diagrams.
- Show what goes into a calculation, not just the final number. For example $|\vec{p}| \approx m|\vec{v}| = (5kg)\times(2\,\tfrac{m}{s}) = 10\,\tfrac{kg\cdot m}{s}$
- Go for partial credit. If you cannot do some portion of a problem, invent a symbol and/or value for the quantity you can't calculate (explain that you are doing this), and use it to do the rest of the problem.
- Each free-response part is worth 10 points.
- You may use your textbook, class notes, and/or Mathematica to solve the problems. If you use Mathematica, make sure you show what goes into the calculation, not just "done on Mathematica." Set the entire problem up and then feel free to evaluate the integrals or the like.
- Not that this will help, but you cannot consult any other texts or the Internet for solutions.
- You are not to consult any other student or instructor with the completion of this exam.
- The exam will be collected electronically. Please email me your solutions as a .pdf file on or before 5pm, Friday, May 30, 2014.

Problem #1	/30
Problem #2	/10
Problem #3	/20
Problem #4	/20
Problem #5	/30
Problem #6	/30
Problem #7	/20
Total	/170

I affirm that I have carried out my academic endeavors with full academic honesty.

____

1. Consider reflection from a step potential of height V_0 with $E > V_0$ but now with an infinitely high wall added at a distance a from the step as shown below.
   a. What is $\psi(x)$ in each region?

The wave functions in each region are given by:

$$x<0: \ \psi(x) = Ae^{ikx} + Be^{-ikx}; \quad k = \frac{\sqrt{2mE}}{\hbar}$$

$$0<x<a: \psi(x) = Ce^{ik'x} + De^{-ik'x}; \quad k' = \frac{\sqrt{2m(E-V_0)}}{\hbar}$$

. Next we impose boundary conditions for the continuity of the wave function and its first derivative at $x = 0$. We have

$$\begin{aligned} &\psi @ x = 0: \quad A + B = C + D \\ &\psi ' @ x = 0: \ ik(A-B) = ik'(C-D) \end{aligned}$$

. Next we have one more boundary condition that the wave function must vanish at $x = a$. We have $\psi = 0 @ x = a: \ Ce^{ik'a} + De^{-ik'a} = 0$. We have three equations in four unknown coefficients so we can express the wave functions in terms of a single unknown amplitude A. Doing this we find:

$$\begin{aligned} A &= \frac{C}{2}\left[1 + \frac{k'}{k} - e^{2ik'a} + \frac{k'}{k}e^{2ik'a}\right] \\ B &= \frac{C}{2}\left[1 - \frac{k'}{k} - e^{2ik'a} - \frac{k'}{k}e^{2ik'a}\right] \\ D &= -Ce^{2ik'a} \end{aligned}$$

. Finishing the wave equation we have:

$$x<0: \psi(x) = \frac{C}{2}\left[1 + \frac{k'}{k} - e^{2ik'a} + \frac{k'}{k}e^{2ik'a}\right]e^{ikx} + \frac{C}{2}\left[1 - \frac{k'}{k} - e^{2ik'a} - \frac{k'}{k}e^{2ik'a}\right]e^{-ikx}$$

$$0<x<a: \ \psi(x) = Ce^{ik'x} - Ce^{2ik'a}e^{-ik'x}$$

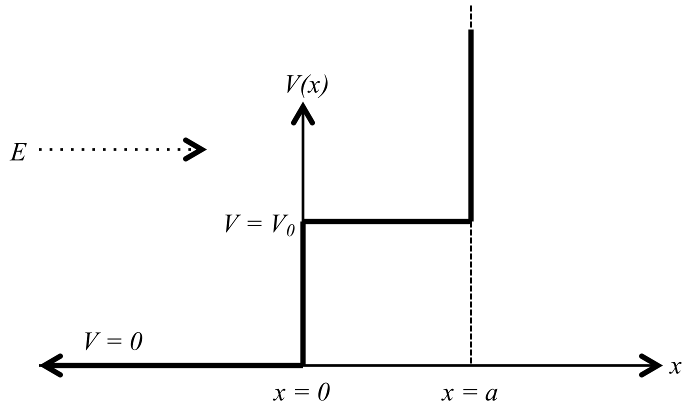
   b. Show that the reflection coefficient at $x = 0$ is $R = 1$. This is different than the previously derived reflection coefficient without the infinite wall? What is the physical reason that $R = 1$ in this case?

$$R = \left(\frac{B}{A}\right)^* \frac{B}{A} = \left(\frac{\frac{C}{2}\left[1 - \frac{k'}{k} - e^{-2ik'a} - \frac{k'}{k}e^{-2ik'a}\right]}{\frac{C}{2}\left[1 + \frac{k'}{k} - e^{-2ik'a} + \frac{k'}{k}e^{-2ik'a}\right]} \right) \times \left(\frac{\frac{C}{2}\left[1 - \frac{k'}{k} - e^{2ik'a} - \frac{k'}{k}e^{2ik'a}\right]}{\frac{C}{2}\left[1 + \frac{k'}{k} - e^{2ik'a} + \frac{k'}{k}e^{2ik'a}\right]} \right) = 1$$

after some easy algebra. The reason why the reflection coefficient has to be identically unity is that what ever may pass the barrier will be reflected back from the infinite wall; so all incident particles will be reflected.

   c. Which part of the wave function represents a leftward moving particle at $x \le 0$? Show that this part of the wave function is an eigenfunction of the momentum operator and calculate the eigenvalue. Is the total wave function for $x \le 0$ an eigenfunction of the momentum operator?

The part of the wave function that represents the left moving particle for $x \le 0$ is given by Be^{-ikx}. To see if this is an eigenfunction of the momentum operator and determine the eigenvalue, we apply the momentum operator. We have $-i\hbar\frac{\partial}{\partial x}\left(Be^{-ikx}\right) = -i\hbar(-ik)Be^{-ikx} = -\hbar k Be^{-ikx}$ therefore the eigenvalue of the momentum operator is $-\hbar k$.



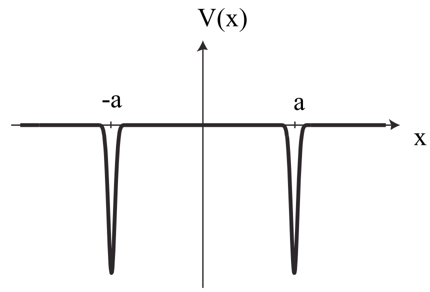
2. A particle moves in one dimension in the potential $V(x) = V_0 \ln\left(\frac{x}{x_0}\right)$ for $x > 0$, where x_0 and V_0 are constants with units of length and energy, respectively. There is an infinite potential barrier located at $x = 0$. The particle drops from the first excited state with energy E_1 into the ground state with energy E_0, by emitting a photon with energy $E_1 - E_0$. Show that the frequency of the photon emitted by this particle is independent of the mass of the particle. (Hint: Define a parameter, s where $s = mx^2$.)

The Schrodinger equation here is given by $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0 \ln\left(\frac{x}{x_0}\right)\psi = E\psi$. Using the hint above, we rewrite the SWE as $-\frac{\hbar^2}{2}\frac{d^2\psi}{ds^2} + \frac{V_0}{2}\ln\left(\frac{s}{s_0}\right)\psi = E\psi$. Splitting up the log functions and introducing a parameter C that has units of $\frac{length^2}{mass}$ we can write $-\frac{\hbar^2}{2}\frac{d^2\psi}{ds^2} + \frac{V_0}{2}\ln\left(\frac{s}{C}\right)\psi = \left(E + \frac{V_0}{2}\ln\left(\frac{s_0}{C}\right)\right)\psi$. The dependence of the energy on mass is through s_0 and changing the mass has the effect of adding the same offset to all the energy levels. Therefore although the energy may depend on mass, the difference in energy levels does not and thus the frequency of the emitted photon does not depend on the mass of the particle.

3. Consider a *double* delta function potential given by $V(x) = -\alpha\left[\delta(x+a)+\delta(x-a)\right]$ where α and a are positive constants.

   a. Sketch this potential and determine the number of bound states it posses. What are the allowed energies, for $\alpha = \frac{\hbar^2}{ma}$ and $\alpha = \frac{\hbar^2}{4ma}$? Sketch the wave function.

The potential looks like,



The solutions split themselves in to even and odd. To determine the number of bound states let's write the wave functions for the even solutions. We have:

$$\psi(x) = \begin{cases} Ae^{-kx} & x > a \\ B\left(e^{kx}+e^{-kx}\right) & -a < x < a \\ Ae^{kx} & x < -a \end{cases}$$

. Next continuity at $x = a$ gives

$Ae^{-ka} = B\left(e^{ka}+e^{-ka}\right) \to A = B\left(e^{2ka}+1\right)$ The derivative is discontinuous, so we use the ideas in problem #5, where $\Delta\left(\frac{d\psi}{dx}\right) = -\frac{2m\alpha}{\hbar^2}\psi(a)$. Evaluating the left hand

$$-kAe^{-ka} - B\left(ke^{ka}+ke^{-ka}\right) = -\frac{2m\alpha}{\hbar^2}Ae^{-ka} \to A + B\left(e^{2ka}-1\right) = \frac{2m\alpha}{\hbar^2 k}A$$

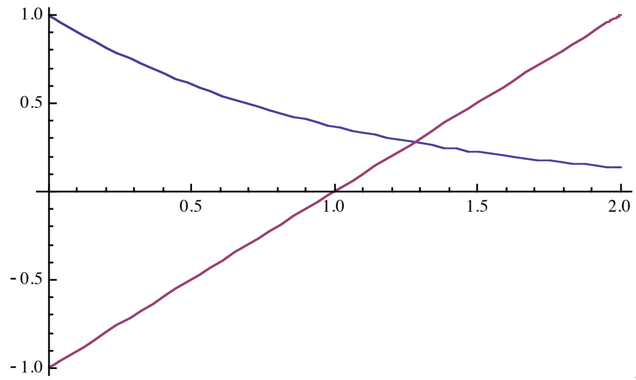
side, we have $B\left(e^{2ka}-1\right) = A\left(\frac{2m\alpha}{\hbar^2 k}-1\right) = B\left(e^{2ka}+1\right)\left(\frac{2m\alpha}{\hbar^2 k}-1\right)$

$$e^{2ka}-1 = e^{2ka}\left(\frac{2m\alpha}{\hbar^2 k}-1\right)+\left(\frac{2m\alpha}{\hbar^2 k}-1\right) \to e^{-2ka} = \left(\frac{\hbar^2 k}{m\alpha}-1\right)$$

This is an equation that's transcendental and must be solved graphically.

Graphing the results in Mathematica, we have, defining $z = 2ka$ and $c = \frac{\hbar^2}{2am\alpha}$ and plotting $e^{-z} = cz-1$, the solution(s) is(are): $z = 1.2785$ using the find root command. Thus for the even solutions we have ONE bound state and the energy of the bound state is given through k and we have,

$$k^2 = -\frac{2mE}{\hbar^2} = \left(\frac{z}{2a}\right)^2 \to E = -\frac{(1.278)^2}{8}\left(\frac{\hbar^2}{ma^2}\right).$$



, where e^{-z} is the blue curve and $cz-1$ is the red curve.

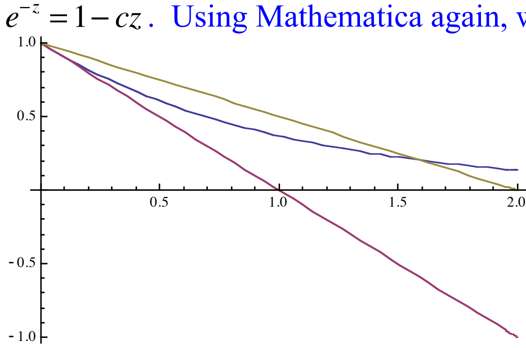
Now let's do the odd solutions. We have for the wave functions for the bound states (if there are any) given by $\psi(x)=\begin{cases} Ae^{-kx} & x>a \\ B\left(e^{kx}-e^{-kx}\right) & -a<x<a \\ -Ae^{kx} & x<-a \end{cases}$. Next continuity at $x=a$ gives $Ae^{-ka}=B\left(e^{ka}-e^{-ka}\right)\to A=B\left(e^{2ka}-1\right)$ The derivative is discontinuous, so we use the ideas in problem #5, where $\Delta\left(\frac{d\psi}{dx}\right)=-\frac{2m\alpha}{\hbar^2}\psi(a)$. Evaluating the left hand side, we have

$$-kAe^{-ka}-B\left(ke^{ka}+ke^{-ka}\right)=-\frac{2m\alpha}{\hbar^2}Ae^{-ka}$$

$$B\left(e^{2ka}+1\right)=A\left(\frac{2m\alpha}{\hbar^2 k}-1\right)=B\left(e^{2ka}-1\right)\left(\frac{2m\alpha}{\hbar^2 k}-1\right)$$

$$e^{2ka}+1=e^{2ka}\left(\frac{2m\alpha}{\hbar^2 k}-1\right)-\frac{2m\alpha}{\hbar^2 k}+1\to e^{-2ka}=\left(1-\frac{\hbar^2 k}{m\alpha}\right)$$

. This is another transcendental equation. Using the same definitions, as before, we will plot $e^{-z}=1-cz$. Using Mathematica again, we have



for the plot. Here there may or may not be a solution. Both graphs have their y-intercepts at 1, but if c is too large (α too small), there may be no intersection (red line), whereas if c is smaller (yellow line) there will be. (Note that $z=0$ which implies that $k=0$ is not a solution, since ψ is then non-normalizable.) So there could be an odd solution for $c<1$ or

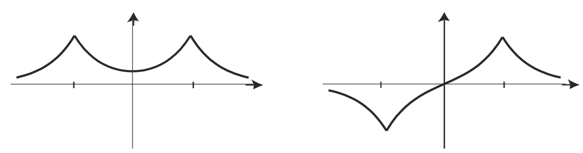
$\alpha > \dfrac{\hbar^2}{2ma}$. So we have one bound state if $\alpha \le \dfrac{\hbar^2}{2ma}$ and two if $\alpha > \dfrac{\hbar^2}{2ma}$. So, for

$$\alpha = \frac{\hbar^2}{ma} \to c = \frac{1}{2} \quad \begin{cases} even:\ e^{-z} = \dfrac{z}{2} - 1 \Rightarrow z = 2.21772 \\ odd:\ e^{-z} = 1 - \dfrac{z}{2} \Rightarrow z = 1.59362 \end{cases}$$

. The energies of the two bound states are given by $E = -0.615\left(\dfrac{\hbar^2}{ma^2}\right)$ or $E = -0.317\left(\dfrac{\hbar^2}{ma^2}\right)$ using the formula for the even solutions. If

$\alpha = \dfrac{\hbar^2}{4ma} \to c = 2 \Rightarrow e^{-z} = 2z - 1 \to z = 0.738835$ and energy $E = -0.0682\left(\dfrac{\hbar^2}{ma^2}\right)$

. The wave functions look like



even odd

b. What is the transmission coefficient for this potential?

The wave functions here are given by: $\psi(x) = \begin{cases} Ae^{-ikx} + Be^{-ikx} & x < -a \\ Ce^{ikx} + De^{-ikx} & -a < x < a \\ Fe^{ikx} & x > a \end{cases}$.

Imposing boundary conditions we have

$\psi @ -a:\ \ Ae^{-ika} + Be^{ika} = Ce^{-ika} + De^{ika}$

$\psi @ +a:\ \ Ce^{ika} + De^{-ika} = Fe^{ika}$

$$disc\,\psi' @ -a : \Delta\left(\frac{d\psi}{dx}\right) = -\frac{2m\alpha}{\hbar^2}\psi(-a):$$

$$ik\left(Ce^{-ika} - De^{ika}\right) - ik\left(Ae^{-ika} - Be^{ika}\right) = -\frac{2m\alpha}{\hbar^2}\left(Ae^{-ika} + Be^{ika}\right).$$

$$disc\,\psi' @ +a : \Delta\left(\frac{d\psi}{dx}\right) = -\frac{2m\alpha}{\hbar^2}\psi(a):$$

$$ikFe^{ika} - ik\left(Ce^{ika} - De^{-ika}\right) = -\frac{2m\alpha}{\hbar^2}Fe^{ika}$$

Next we follow the example for the finite potential barrier in the homework. After a lot of algebra the transmission coefficient is given by

$$T=\left|\frac{F}{A}\right|^2=\frac{8\left(\frac{\hbar^2 k}{2m\alpha}\right)^4}{8\left(\frac{\hbar^2 k}{2m\alpha}\right)^4+4\left(\frac{\hbar^2 k}{2m\alpha}\right)^2+\left(4\left(\frac{\hbar^2 k}{2m\alpha}\right)^2-1\right)\cos(4ka)-4\left(\frac{\hbar^2 k}{2m\alpha}\right)\sin(4ka)}$$

4. Suppose that you have a particle of mass m moving in a harmonic oscillator potential.
   a. Starting from the ground state, what is the energy and wave function for the 5th excited state the using the raising and/or lowering operators?
   b. What are the expectation value of the potential and kinetic energies for this state? Are they as expected?

   See the external .pdf file associated with this solution.

5. Consider a particle of mass m in a one-dimensional potential $V(x)=-\frac{\hbar^2}{2m}P\delta(x)$, where P is a positive quantity and $\delta(x)$ is the Dirac delta function.

   a. Show that the dimension of P is inverse length?

$$[\hbar]=J\cdot s$$
$$[m]=kg$$
$$[\delta]=m^{-1}$$
$$[V]=J$$
$$\therefore J=\left(\frac{J^2\cdot s^2}{kg\cdot m}\right)[P]\rightarrow[P]=\frac{J\cdot kg\cdot m}{J^2\cdot s^2}=\frac{kg\cdot m\cdot s^2}{kg\cdot m^2\cdot s^2}=\frac{1}{m}$$

   b. Show that for this potential, an eigenfunction $\psi(x)$ satisfies $\psi'(0-)-\psi'(0+)=P\psi(0)$, where $\psi'(0-)$ and $\psi'(0+)$ are the values of the derivatives of the eigenfunction immediately on the left and right, respectively, of the delta function potential and $\psi(0)$ is the value of the eigenfunction at $x=0$.

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}[E - V]\psi$$

$$\int_{-\varepsilon}^{\varepsilon} \frac{d^2\psi}{dx^2}dx = \left.\frac{d\psi}{dx}\right|_{-\varepsilon}^{\varepsilon} = \psi'(\varepsilon) - \psi'(-\varepsilon) = -\frac{2m}{\hbar^2}\int_{-\varepsilon}^{\varepsilon} E\psi\, dx + \frac{2m}{\hbar^2}\int_{-\varepsilon}^{\varepsilon} V\psi\, dx$$

$$\psi'(\varepsilon) - \psi'(-\varepsilon) = 0 - \frac{2m\hbar^2}{2m\hbar^2}\int_{-\varepsilon}^{\varepsilon} P\delta(x)\psi\, dx = -P\psi(0)$$

$$\therefore P\psi(0) = \psi'(-\varepsilon) - \psi'(\varepsilon) = \psi'(0-) - \psi'(0+)$$

c. What is the energy for a bound state?

The energy of the bound state is given by solving the SWE.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \frac{\hbar^2}{2m}P\delta(x)\psi = E\psi$$

Bound states: $E < 0$

$$\frac{d^2\psi}{dx^2} + P\delta(x)\psi = \frac{2mE}{\hbar^2}\psi = k^2\psi$$

The eigenfunctions must vanish at $x = \pm\infty$, so we have solutions

$$\psi = \begin{cases} Ae^{-kx} & x > 0 \\ Be^{kx} & x < 0 \end{cases}$$

Next we have that the wave function is continuous at $x = 0$ and this produces $\psi(0-) = \psi(0+) \to B = A$.

Then we take the derivatives of the wave function above and below $x = 0$ and then use the results from part b.

$$\psi'(x-) = Ake^{kx} \to \psi'(0-) = Ak$$

$$\psi'(x+) = -Ake^{-kx} \to \psi'(0+) = -Ak$$

$$\therefore \psi'(0-) - \psi'(0+) = P\psi(0) \to 2Ak = PA$$

$$k = \frac{P}{2} \to k^2 = \frac{P^2}{4} = \frac{2mE}{\hbar^2} \to E = \frac{\hbar^2 P^2}{8m}$$

6. An electron (mass m and charge e) is confined inside a hollow sphere of radius a and the spherical wall is impenetrable.

   a. What is the ground state energy?

So the solution starts with the SWE in spherical coordinates. However, I'm going to take a shortcut and start with the radial equation. The radial equation looks like $\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\frac{2mr^2}{\hbar^2}(V-E)=l(l+1)$. Making the change of variables $R=\frac{u}{r}$ we have for the derivatives $\frac{dR}{dr}=\frac{d}{dr}\left(\frac{u}{r}\right)=-\frac{u}{r^2}+\frac{1}{r}\frac{du}{dr}$ and multiply through by r^2 gives $r^2\frac{dR}{dr}=-u+r\frac{du}{dr}$. Now take the next derivative. We get $\frac{d}{dr}\left(-u+r\frac{du}{dr}\right)=r\frac{d^2u}{dr^2}$. Inserting this expression into the SWE we get $-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\left[V+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u=Eu$. In the ground state $n=1$ and $l=0$, so we have $-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+Vu=Eu$, where $V=\begin{cases}0 & r<a\\ \infty & r>a\end{cases}$ and we want to look for the electron in a region $r<a$, so the radial wave equation reduces to $\frac{d^2u}{dr^2}=-\frac{2mE}{\hbar^2}u=-k^2u\rightarrow u=A\sin(kr)+B\cos(kr)$. Next we want the wave function to be finite as $r\rightarrow\infty$ so $B=0$ and the solution is $u=A\sin(kr)$. IN terms of R, $R=\frac{u}{r}=\frac{A\sin(kr)}{r}$. Lastly, to determine the energy, we note at $r=a,\ u=0=A\sin(ka)\rightarrow ka=n\pi\rightarrow k^2=\frac{2mE}{\hbar^2}=\frac{n^2\pi^2}{a^2}$.

$$\therefore E_{n=1}=\frac{\hbar^2\pi^2}{2ma^2}$$

b. What is the ground state wave function?

The ground state wave function is given by $R=\frac{A\sin(kr)}{r}=\frac{A\sin\left(\frac{n\pi}{a}r\right)}{r}$.

Normalizing the solution (where for the ground state $n=1$) we have, $1=\int_0^\infty A^2\frac{\sin^2\left(\frac{\pi r}{a}\right)}{r^2}dr\rightarrow 1=A^2\left(\frac{\pi^2}{2a}\right)\rightarrow A=\frac{\sqrt{2a}}{\pi}$. Thus the ground state wave function $\psi_0(r)=\frac{\sqrt{2a}}{\pi}\frac{\sin\left(\frac{\pi r}{a}\right)}{r}$.

c. What is the first excited state energy and wave function?

To determine the wave functions we return to the SWE above, where $-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\left[V+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u=Eu$. Multiplying through by $-\frac{2m}{\hbar^2}$ we get $\frac{d^2u}{dr^2}=\left[\frac{l(l+1)}{r^2}-\frac{2mE}{\hbar^2}\right]u=\left[\frac{l(l+1)}{r^2}-k^2\right]u$ where for $r<a, V=0$. The general solutions to this are given in the hint below, and they are spherical Bessel or Neumann functions. But, the Neumann functions blow up as $r\to 0$ we choose the Bessel functions. And the boundary condition at $r=a$ gives $u(a)=0=Aaj_l(ka)\to j_l(ka)=0\to k=\frac{1}{a}\beta_{nl}^2$ where the solutions are computed numerically and β_{nl}^2 is the n^{th} zero of the l^{th} Bessel function. The energies are given by $E_{nl}=\frac{\hbar^2}{2ma}\beta_{nl}^2$. For this problem let's look at $(n,l)=(1,0)$. For $l=0$ we need the zeroth order Bessel function, $j_o=\frac{\sin x}{x}$. Therefore, $R(r)=\frac{u}{r}=Aj_o(kr)=A\frac{\sin(kr)}{kr}$. Normalizing we have $1=\int_0^\infty A^2\frac{\sin^2(kr)}{k^2r^2}dr\to 1=A^2\left(\frac{\pi}{2k}\right)\to A=\sqrt{\frac{2k}{\pi}}$. Thus the first excited state wave function is $R(r)=\sqrt{\frac{2k}{\pi}}\left(\frac{\sin(kr)}{kr}\right)$.

Evaluating the energy we need the first zero of the zeroth Bessel function. From Mathematica, we have $\beta=2.40483$ using the BesselJZero command, and the energy is then $E_{10}=\frac{\hbar^2}{2ma}(2.40483)^2\sim\frac{3\hbar^2}{ma}$.

Hint: The following may be useful. The general solution to the equation $-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\left[V+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u=Eu$ is $u(r)=Arj_l(kr)+Brn_l(kr)$, where $j_l(x)=(-1)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\frac{\sin x}{x}$ are the spherical Bessel functions of order l and $n_l(x)=-(-1)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\frac{\cos x}{x}$ are the spherical Neumann functions of order l.

7. The solution for the Schrodinger equation for the ground state of a hydrogen atom is given by $\psi_0=\frac{1}{\sqrt{\pi a_0^3}}e^{-\frac{r}{a_0}}$ where, a_0 is the Bohr radius and r is the distance from the origin.

a. What is the most probable value of r to find the electron?

From the probability or radial density function we have

$4\pi r^2 |\psi_0|^2 = 4\pi r^2 \left| \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}} \right|^2 = \frac{4\pi r^2}{\pi a_0^3} e^{-\frac{2r}{a_0}}$. The most probable value is where this is a maximum. So we'll take the derivative of this expression and set it equal to zero and solve for r.

$$\frac{d}{dr}\left(\frac{4r^2}{a_0^3} e^{-\frac{2r}{a_0}} \right) = \frac{4}{a_0^3}\left(2re^{-\frac{2r}{a_0}} - \frac{2}{a_0} r^2 e^{-\frac{2r}{a_0}} \right) = 0$$

$$1 - \frac{r}{a_0} = 0 \rightarrow r = a_0$$

b. What is the expectation value of the position?

The expectation value is given in the usual way:

$$\langle r \rangle = \int \psi^* r \psi r^2 \, dr = \tfrac{1}{\pi a_0^3} \int_0^\infty r^3 e^{-\frac{2r}{a_0}} \, dr = \tfrac{1}{\pi a_0^3} \left[\frac{6a_0^4}{16} \right] = \frac{6a_0}{16\pi} = 1.18a_0$$