Physics 220

Exam #2

May 20, 2016

Name___________________________________

Please read and follow these instructions carefully:

- Read all problems carefully before attempting to solve them.
- Your work must be legible, and the organization clear.
- You must show all work, including correct vector notation.
- You will not receive full credit for correct answers without adequate explanations.
- You will not receive full credit if incorrect work or explanations are mixed in with correct work. So erase or cross out anything you don't want graded.
- Make explanations complete but brief. Do not write a lot of prose.
- Include diagrams.
- Show what goes into a calculation, not just the final number. For example $|\vec{p}| \approx m|\vec{v}| = (5kg)\times(2\,\frac{m}{s}) = 10\,\frac{kg\cdot m}{s}$
- Give standard SI units with your results unless specifically asked for a certain unit.
- Unless specifically asked to derive a result, you may start with the formulas given on the formula sheet including equations corresponding to the fundamental concepts.
- Go for partial credit. If you cannot do some portion of a problem, invent a symbol and/or value for the quantity you can't calculate (explain that you are doing this), and use it to do the rest of the problem.
- Each free-response part is worth 20 points

Problem #1	/60
Problem #2	/60
Total	/120

I affirm that I have carried out my academic endeavors with full academic honesty.

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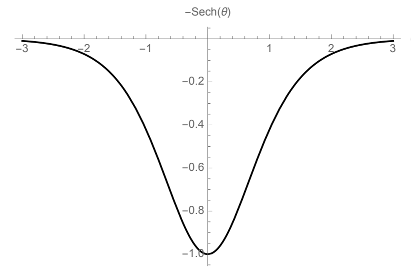
1. Consider the potential $V(x) = -\frac{\hbar^2 a^2}{m}\text{sech}^2(ax)$, where a is a positive constant and $\text{sech}(\theta)$ is the hyperbolic secant function. The potential is graphed on the right.



a. Show that the ground state wave function can be written as $|\psi_0\rangle = A\,\text{sech}(ax)$. Determine the energy of the ground state E_0 and the normalization constant A.

Hints: You may need $\tanh^2\theta + \text{sech}^2\theta = 1$, $\frac{d}{d\theta}\cosh\theta = \sinh\theta$, $\frac{d}{d\theta}\sinh\theta = \cosh\theta$, and $\int_{-\infty}^{\infty}\text{sech}^2(ax)dx = \frac{2}{a}$.

The Schrödinger equation is $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$. We need to determine $\frac{d^2\psi}{dx^2}$.

$$\frac{d\psi}{dx} = A\frac{d}{dx}\left(\text{sech}(ax)\right) = A\frac{d}{dx}\left(\cosh^{-1}(ax)\right) = -Aa\cosh^{-2}(ax)\sinh(ax) = -a\tanh(ax)\psi$$

$$\frac{d^2\psi}{dx^2} = \frac{d}{dx}\left(\frac{d\psi}{dx}\right) = \frac{d}{dx}\left(-a\tanh(ax)\psi\right) = -a\left\{\psi\frac{d}{dx}\left(\tanh(ax)\right) + \tanh(ax)\frac{d\psi}{dx}\right\}$$

$$\frac{d^2\psi}{dx^2} = -a\left\{-a\left(\tanh^2(ax) - 1\right) - a\tanh^2(ax)\right\}\psi = \left(2\tanh^2(ax) - 1\right)a^2\psi$$

Now the Schrödinger wave equation says

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

$$-\frac{\hbar^2 a^2}{2m}\left(2\tanh^2(ax) - 1\right)\psi - \frac{\hbar^2 a^2}{m}\text{sech}^2(ax)\psi = E\psi$$

$$-\frac{2\hbar^2 a^2}{2m}\left(1 - \text{sech}^2(ax)\right) + \frac{\hbar^2 a^2}{2m} - \frac{\hbar^2 a^2}{m}\text{sech}^2(ax) = E$$

$$-\frac{\hbar^2 a^2}{m} + \frac{\hbar^2 a^2}{2m} = E$$

$$E = -\frac{\hbar^2 a^2}{2m}$$

And we see that ψ is a solution to the Schrödinger wave equation with energy $E = -\frac{\hbar^2 a^2}{2m}$.

b. Suppose that you have the wave function given as $|\psi_k(x)\rangle = A\left(\frac{ik - a\tanh(ax)}{\beta}\right)e^{ikx}$ where $k = \sqrt{\frac{2mE}{\hbar^2}}$ and $\beta = ik + a$. Does this wave function correspond to a state of definite energy? If so, what is the energy? If not, explain why it does not.

The Schrödinger equation is $\hat{H}|\psi\rangle = \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V\right)|\psi\rangle = E|\psi\rangle$. We need to determine $\frac{d^2\psi}{dx^2}$.

$$|\psi_k\rangle = A\left(\frac{ik - a\tanh(ax)}{\beta}\right)e^{ikx} = \frac{ikA}{\beta}\left(e^{ikx} + \frac{ia}{k}e^{ikx}\tanh ax\right)$$

$$\frac{d}{dx}|\psi_k\rangle = \frac{ikA}{\beta}\left(ike^{ikx} - ae^{ikx}\tanh ax + \frac{ia^2}{k}e^{ikx}\operatorname{sech}^2 ax\right)$$

where we evaluated

$$\frac{d}{dx}(\tanh ax) = \frac{d}{dx}\left(\sinh(ax)\cosh^{-1}(ax)\right) = a\left[1 - \left(\sinh^2(ax)\cosh^{-2}(ax)\right)\right] = a\left[1 - \tanh^2 ax\right]$$

$$\frac{d}{dx}(\tanh ax) = a\operatorname{sech}^2 ax$$

The second derivative:

$$\frac{d^2}{dx^2}|\psi_k\rangle = \frac{d}{dx}\left(\frac{d}{dx}|\psi_k\rangle\right) = \frac{ikA}{\beta}\frac{d}{dx}\left(ike^{ikx} - ae^{ikx}\tanh ax + \frac{ia^2}{k}e^{ikx}\operatorname{sech}^2 ax\right)$$

$$= \frac{ikA}{\beta}\left[-k^2e^{ikx} + \frac{ia^2}{k}\left(ike^{ikx}\operatorname{sech}^2 ax\right) + \frac{ia^2}{k}e^{ikx}\left(-2a\tanh ax\operatorname{sech}^2 ax\right) - ikae^{ikx}\tanh ax - a^2e^{ikx}\operatorname{sech}^2 ax\right]$$

$$\frac{d^2}{dx^2}|\psi_k\rangle = \frac{ikA}{\beta}e^{ikx}\left[-k^2 - 2a^2\operatorname{sech}^2 ax - \frac{2ia^3}{k}\tanh ax\operatorname{sech}^2 ax - ika\tanh ax\right]$$

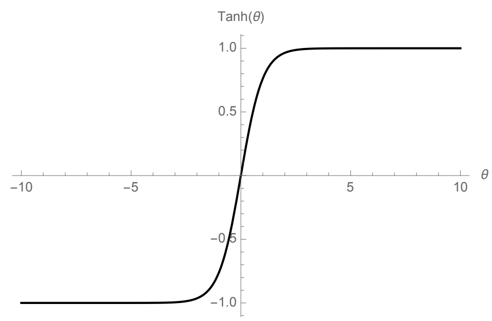
Now the wave equation looks like, factoring out all of the common terms:

$$-\frac{\hbar^2 ikA}{\beta m}e^{ika}\left[-\frac{k^2}{2} - a^2\operatorname{sech}^2 ax - \frac{ia^3}{k}\tanh ax\operatorname{sech}^2 ax - \frac{ika}{2}\tanh ax + a^2\operatorname{sech}^2 ax + \frac{ia^3}{k}\tanh ax\operatorname{sech}^2 ax\right]$$

$$= \frac{\hbar^2 ik^2A}{2\beta m}e^{ika}\left[k + ia\tanh ax\right] = \frac{\hbar^2k^2}{2m}\left[A\left(\frac{ik - a\tanh ax}{\beta}\right)e^{ika}\right]$$

Therefore, $\hat{H}|\psi_k\rangle = \frac{\hbar^2k^2}{2m}|\psi_k\rangle = E|\psi_k\rangle$ and is a state of definite energy with energy $\frac{\hbar^2k^2}{2m}$.

The graph below shows the hyperbolic tangent function as a function of theta. Using the graph, show that for large values of negative x that $\psi_k(x) = A\left(\frac{ik - a\tanh(ax)}{ik+a}\right)e^{ikx} \approx Ae^{ikx}$ and also determine the asymptotic form for $\psi_k(x)$ for large positive x. From your results, determine the transmission and reflection coefficients and comment on the result.



For large negative or positive x, we have $\psi_k(x) = A\left(\frac{ik - a\tanh(ax)}{ik+a}\right)e^{ikx}$. From the graph, we see that as $x \to -\infty$, the hyperbolic tangent function goes to negative one. Thus, $\psi_k(x) = A\left(\frac{ik - a\tanh(ax)}{ik+a}\right)e^{ikx} \sim A\left(\frac{ik+a}{ik+a}\right)e^{ikx} \sim Ae^{ikx}$. This represents a wave coming into the barrier from the left with no accompanying reflected wave.

Further, from the graph, we see that as $x \to +\infty$, the hyperbolic tangent function goes to positive one. Thus, $\psi_k(x) = A\left(\frac{ik - a\tanh(ax)}{ik+a}\right)e^{ikx} \sim A\left(\frac{ik-a}{ik+a}\right)e^{ikx}$. This represents a transmitted wave on the right of the barrier.

The transmission coefficient:

$$T = \left|\left(\frac{\text{Amplitude of outgoing wave}}{\text{Amplitude of incoming wave}}\right)^2\right| = \frac{\left(A^*\left(\frac{-ik-a}{-ik+a}\right)e^{-ikx}\right)\left(A\left(\frac{ik-a}{ik+a}\right)e^{ikx}\right)}{A^*A} = 1.$$

The reflection coefficient: $R = 1 - T = 0$.

This particular potential produces no reflections (a reflectionless potential) and every particle regardless of its energy passes.

2. Suppose that an electron in a hydrogen atom is in the state $|\psi_{211}\rangle$.

a. What is the probability of finding the electron inside of the nucleus of hydrogen of radius b? Express your answer in terms of a and b.

Hints:

1. The radial portion of the wave function is $R_{21}(r) = \frac{1}{\sqrt{24a^5}} r e^{-\frac{r}{2a}}$.

2. $\int x^n e^{bx}\, dx = \frac{e^{bx}}{b}\left[x^n - \frac{nx^{n-1}}{b} + \frac{n(n-1)x^{n-2}}{b^2} - \ldots \frac{(-1)^n n!}{b^n} \right]$

3. The Taylor expansion of the exponential function is $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$.

4. Assume that $b << a$.

We must first determine $|\psi_{211}\rangle = R_{21} Y_{11}$.

$$Y_l^{m_l}(\theta,\phi) = \varepsilon \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-|m_l|)!}{(l+|m_l|)!}} e^{im\phi} P_l^{m_l}(\cos\theta); \quad \varepsilon = \begin{cases} (-1)^{m_l} & m_l \geq 0 \\ 1 & m_l \leq 0 \end{cases}$$

$$Y_1^1(\theta,\phi) = (-1)^1 \sqrt{\frac{(2(1)+1)}{4\pi} \frac{(1-|1|)!}{(1+|1|)!}} e^{i\phi} P_1^1(\cos\theta) = -\sqrt{\frac{3}{8\pi}} e^{i\phi} P_1^1(\cos\theta)$$

where,

$$P_l^{m_l}(\cos\theta) = (1-\cos^2\theta)^{\frac{|m_l|}{2}} \left(\frac{d}{d\cos\theta} \right)^{|m_l|} P_l(\cos\theta)$$

$$P_1^1(\cos\theta) = (1-\cos^2\theta)^{\frac{|1|}{2}} \frac{d}{d\cos\theta}(P_1(\cos\theta))$$

and where,

$$P_l(\cos\theta) = \frac{1}{2^l l!} \left(\frac{d}{d\cos\theta} \right)^l (\cos^2\theta - 1)^l$$

$$P_1(\cos\theta) = \frac{1}{2^1 1!} \frac{d}{d\cos\theta} (\cos^2\theta - 1) = \cos(\theta)$$

Thus,

$$P_1^1(\cos\theta) = (1-\cos^2\theta)^{\frac{|1|}{2}} \frac{d}{d\cos\theta}(P_1(\cos\theta)) = \sin\theta \frac{d}{d\cos\theta}(\cos\theta) = \sin\theta.$$

Therefore, $Y_1^1(\theta,\phi) = -\sqrt{\frac{3}{8\pi}} e^{i\phi} P_1^1(\cos\theta) = -\sqrt{\frac{3}{8\pi}} e^{i\phi} \sin\theta$.

Now we can form the state $|\psi_{211}\rangle$.

$$\left|\psi_{211}\right\rangle = R_{21}(r)Y_1^1(\theta,\phi) = \frac{1}{\sqrt{24a^5}} re^{-\frac{r}{2a}}\left(-\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta\right)$$

$$\left|\psi_{211}\right\rangle = -\sqrt{\frac{3}{24\cdot 8\pi a^5}} re^{-\frac{r}{2a}}e^{i\phi}$$

$$\left|\psi_{211}\right\rangle = -\frac{1}{\sqrt{\pi a}}\frac{r}{8a^2}e^{-\frac{r}{2a}}\sin\theta e^{i\phi}$$

The probability is given in the usual way:

$$\left\langle\psi_{211}\right| = -\frac{1}{\sqrt{\pi a}}\frac{r}{8a^2}e^{-\frac{r}{2a}}\sin\theta e^{-i\phi}$$

$$P = \left\langle\psi_{211}\middle|\psi_{211}\right\rangle = \int\psi^*\psi d^3r = \frac{1}{64\pi a^5}\int_0^{2\pi}d\phi\int_0^{\pi}\sin^3\theta\, d\theta\int_0^{b}r^4e^{-\frac{r}{a}}\,dr$$

$$\int_0^{2\pi}d\phi = 2\pi$$

$$\int_0^{\pi}\sin^3\theta\, d\theta = \left[-\frac{3\cos[\theta]}{4}+\frac{\cos[3\theta]}{12}\right]_0^{\pi}$$

$$= \left(\left[-\frac{3\cos\pi}{4}+\frac{\cos 3\pi}{12}\right]-\left[-\frac{3\cos 0}{4}+\frac{\cos 0}{12}\right]\right)$$

$$= \left(+\frac{3}{4}-\frac{1}{12}\right)-\left(-\frac{3}{4}+\frac{1}{12}\right) = \frac{6}{4}-\frac{1}{6} = \frac{36-4}{24} = \frac{32}{24} = \frac{4}{3}$$

$$\int_0^{b}r^4e^{-\frac{r}{a}}\,dr = -ae^{-\frac{r}{a}}\left[r^4+4ar^3+12a^2r^2+24a^3r+24a^4\right]_0^{b}$$

$$= -e^{-\frac{b}{a}}\left(\left[ab^4+4a^2b^3+12a^3b^2+24a^4b+24a^5\right]-e^0\left[-24a^5\right]\right)$$

$$= 24a^5 - a^5\left[\frac{b^4}{a^4}+\frac{4b^3}{a^3}+\frac{12b^2}{a^2}+\frac{24b}{a}+24\right]e^{-\frac{b}{a}}$$

$$\approx 24a^5 - a^5\left[\frac{24b}{a}+24\right]e^{-\frac{b}{a}}$$

$$P = \frac{2\pi}{64\pi a^5}\left(\frac{4}{3}\right)\left(24a^5 - a^5\left[\frac{24b}{a}+24\right]e^{-\frac{b}{a}}\right) = \left(1-\left(\frac{b}{a}+1\right)\left(1-\frac{b}{a}\right)\right)$$

$$P = 1-\left(-\frac{b^2}{a^2}+1\right)$$

$$P = \frac{b^2}{a^2}$$

b. What is the expectation value of $\langle r\rangle$ for the electron in the state $|\psi_{211}\rangle$? Express your answer in terms of a. Hint: You'll need a hint from part a.

$$\langle r\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left(\psi^* r\psi\right)r^2\,dr\sin\theta\,d\theta\,d\varphi = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left(\frac{1}{64a^5\pi}e^{-\frac{r}{a}}\sin^2\theta\right)r^5\,dr\sin\theta\,d\theta\,d\phi$$

$$\langle r\rangle = \frac{1}{64a^5\pi}\int_0^{\infty}r^5e^{-\frac{r}{a}}\,dr\int_0^{\pi}\sin^3\theta\,d\theta\int_0^{2\pi}d\phi$$

$$\langle r\rangle = \frac{1}{64a^5\pi}\left[120a^6\right]\left[\frac{4}{3}\right]\left[2\pi\right]$$

$$\langle r\rangle = 5a$$

The radial integral evaluates to:

$$\int_0^{\infty}r^5e^{-\frac{r}{a}}\,dr = ae^{-\frac{r}{a}}\left[r^5-5ar^4+20a^2r^3-60a^3r^2+120a^4-120a^5\right]_0^{\infty}$$

$$\int_0^{\infty}r^5e^{-\frac{r}{a}}\,dr = 0-\left(-120a^6\right)=120a^6$$

and the theta integral was evaluated in part a.

c. Determine $L_z|\psi_{211}\rangle$ and the associated eigenvalue.

$$L_z|\psi_{211}\rangle = -i\hbar\frac{\partial}{\partial\phi}\left\{\frac{1}{\sqrt{a\pi}}\frac{1}{8a^2}re^{-\frac{r}{2a}}\sin\theta e^{i\phi}\right\} = \hbar|\psi_{211}\rangle \text{ and the eigenvalue is } \hbar.$$

Physics 220 Equations

Useful Integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int \cos^2(qx)dx = \frac{x}{2} + \frac{\sin[2qx]}{4q}$$

$$\int \sin^2(qx)dx = \frac{x}{2} - \frac{\sin[2qx]}{4q}$$

$$\int \cos^3(qx)dx = \frac{3\sin[qx]}{4q} + \frac{\sin[3qx]}{12q}$$

$$\int \sin^3(qx)dx = -\frac{3\cos[qx]}{4q} + \frac{\cos[3qx]}{12q}$$

$$\int_{-a/2}^{a/2} x\cos^2(qx)dx = 0$$

$$\int_{-a/2}^{a/2} x\sin^2(qx)dx = 0$$

$$\int_{-a/2}^{a/2} \sin(qx)\cos(qx)dx = 0$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a}$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{\infty} xe^{-ax^2}\,dx = 0$$

$$\int_{-\infty}^{\infty} x^2e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{2a^{\frac{3}{2}}}$$

$$\int_{400nm}^{700nm} \frac{2\pi hc^2}{\lambda^5}\left[\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right]d\lambda = 1197\,\tfrac{W}{m^2}$$

Constants:

$g = 9.8\,\tfrac{m}{s^2}$

$G = 6.67\times10^{-11}\,\tfrac{Nm^2}{kg^2}$

$c = 3\times10^{8}\,\tfrac{m}{s}$

$\sigma = 5.67\times10^{-8}$

$k_B = 1.38\times10^{-23}\,\tfrac{J}{K}$

$1eV = 1.6\times10^{-19}\,J$

$1e = 1.6\times10^{-19}\,C$

$h = 6.63\times10^{-34}\,Js;$

$m_e = 9.11\times10^{-31}\,kg = 0.511\,\tfrac{MeV}{c^2}$

$m_p = 1.67\times10^{-27}\,kg = 938\,\tfrac{MeV}{c^2}$

$m_n = 1.69\times10^{-27}\,kg = 939\,\tfrac{MeV}{c^2}$

$m_E = 6\times10^{24}\,kg$

$R_E = 6.4\times10^{6}\,m$

$a = 0.5\times10^{-10}\,m$

Formulas :

$$c = \upsilon\lambda$$

$$E = h\upsilon = \frac{hc}{\lambda}$$

$$\frac{dS}{d\lambda} = \frac{2\pi hc^2}{\lambda^5}\left[\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right]$$

$$\frac{dS}{d\upsilon} = \frac{2\pi h\upsilon^3}{c^2}\left(\frac{1}{e^{\frac{h\upsilon}{kT}}-1}\right)$$

$$\frac{dS}{d\lambda} = \frac{2\pi ckT}{\lambda^4}$$

$$\lambda_{\max} = \frac{2.9\times10^{-3}\,m\cdot K}{T}$$

$$S = \sigma T^4$$

$$eV_{stop} = hf - \phi$$

$$\lambda' = \lambda + \frac{h}{mc}(1-\cos\phi)$$

$$\hbar = \frac{h}{2\pi};\ k = \frac{2\pi}{\lambda};\ \omega = 2\pi f$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t} = E\psi$$

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

$$\hat{p} = -i\hbar\frac{\partial}{dx}$$

$$\hat{T} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V$$

$$\hat{x} = x$$

$$\langle O\rangle = \int \psi^*\hat{O}\psi\,dr$$

$$P = \int \psi^*\psi\,dx$$

$$E_n = n^2\left(\frac{\pi^2\hbar^2}{2ma^2}\right)$$

$$\Psi_n(x,t) = \sqrt{\frac{2}{a}}\sin(k_nx)e^{-i\frac{E_n}{\hbar}t}$$

$$H_n(q) = (-1)^n e^{q^2} \left(\frac{d}{dq} \right)^n e^{-q^2}; \quad q = \sqrt{\tfrac{m\omega}{2\hbar}} x$$

$$Y_l^{m_l}(\theta,\phi) = \varepsilon \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-|m_l|)!}{(l+|m_l|)!}} e^{im\phi} P_l^{m_l}(\cos\theta); \quad \varepsilon = \begin{cases} (-1)^{m_l} & m_l \geq 0 \\ 1 & m_l \leq 0 \end{cases}$$

$$P_l^{m_l}(\cos\theta) = \left(1-\cos^2\theta\right)^{\frac{|m_l|}{2}} \left(\frac{d}{d\cos\theta} \right)^{|m_l|} P_l(\cos\theta)$$

$$P_l(\cos\theta) = \frac{1}{2^l l!} \left(\frac{d}{d\cos\theta} \right)^l \left(\cos^2\theta - 1\right)^l$$

$$L_{n-l-1}^{2l+1}\left(\tfrac{2r}{na}\right) = (-1)^{2l+1} \left(\tfrac{na}{2}\right)^{2l+1} \left(\tfrac{d}{dr}\right)^{2l+1} L_{n+2l}\left(\tfrac{2r}{na}\right)$$

$$L_{n+2l}\left(\tfrac{2r}{na}\right) = e^{\frac{2r}{na}} \left(\tfrac{na}{2}\right)^{n+2l} \left(\tfrac{d}{dr}\right)^{n+2l} \left(e^{-\frac{2r}{na}} \left(\tfrac{2r}{na}\right)^{n+2l} \right)$$

$$\left|\psi_{nlm_l}\right\rangle = \sqrt{\left(\tfrac{2}{na}\right)^3 \frac{(n-l-a)!}{2n\left[(n+l)!\right]^3}} e^{-\frac{2r}{na}} \left(\tfrac{2r}{na}\right)^l \left[L_{n-l-1}^{2l+1}\left(\tfrac{2r}{na}\right) \right] Y_l^{m_l}(\theta,\phi)$$

$$a_\pm = \frac{1}{\sqrt{2m\hbar\omega}} (\mp ip + m\omega x)$$

$$H = \left(a_\pm a_\mp \pm \tfrac{1}{2}\right)\hbar\omega$$

$$L_\pm = L_x \pm iL_y$$

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

$$L_z = -i\hbar \frac{\partial}{\partial\phi}$$

$$P = \int \psi^* \psi \, d^3r = \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \psi^* \psi \, r^2 \, dr \sin\theta \, d\theta d\phi$$