Physics 220

Exam #2

May 19, 2017

Name___________________________________

Please read and follow these instructions carefully:

- Read all problems carefully before attempting to solve them.
- Your work must be legible, and the organization clear.
- You must show all work, including correct vector notation.
- You will not receive full credit for correct answers without adequate explanations.
- You will not receive full credit if incorrect work or explanations are mixed in with correct work. So erase or cross out anything you don't want graded.
- Make explanations complete but brief. Do not write a lot of prose.
- Include diagrams.
- Show what goes into a calculation, not just the final number. For example $|\vec{p}| \approx m|\vec{v}| = (5kg)\times(2\,\frac{m}{s}) = 10\,\frac{kg\cdot m}{s}$
- Give standard SI units with your results unless specifically asked for a certain unit.
- Unless specifically asked to derive a result, you may start with the formulas given on the formula sheet including equations corresponding to the fundamental concepts.
- Go for partial credit. If you cannot do some portion of a problem, invent a symbol and/or value for the quantity you can't calculate (explain that you are doing this), and use it to do the rest of the problem.
- Each free-response part is worth 10 points.
- All Mathematica calculations need to be commented and printed out to earn full credit.

Problem #1	/40
Problem #2	/40
Total	/80

I affirm that I have carried out my academic endeavors with full academic honesty.

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1. Supose that a particle of mass m is in the first excited state of a harmonic osciallator potential, $V = \frac{m\omega^2}{2}x^2$.

   a. By using the raising and lowering operators, what are $\langle T \rangle$ and $\langle V \rangle$ for this state? Hints: The wave functions for the harmonic oscillator are given by $|\psi_n\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega}{2\hbar}x^2}$, where $H_n(q)$ is a Hermite polynomial of order n. In addition the raising and lowering operators are defined by $a_\pm = \frac{1}{\sqrt{2m\hbar\omega}}(\mp ip + m\omega x)$, and you may need $a_+|\psi_n\rangle = \sqrt{n+1}|\psi_{n+1}\rangle$ and $a_-|\psi_n\rangle = \sqrt{n}|\psi_{n-1}\rangle$.

   Space for calculation of $\langle T \rangle$.

   We start by expressing the momentum and position operators in terms of the raising and lowering operators. We have $a_+ = \frac{1}{\sqrt{2m\hbar\omega}}(-ip + m\omega x)$ and $a_- = \frac{1}{\sqrt{2m\hbar\omega}}(+ip + m\omega x)$. Subtracting a_- from a_+ we can form an expression for p and we have $p = \frac{\sqrt{2m\hbar\omega}}{2} i(a_+ - a_-)$. To determine the expectation value of the kinetic energy, we will need to square the momentum operator and form the kinetic energy operator. Squaring we have:
   $T = \frac{p^2}{2m} = -\frac{2m\hbar\omega}{8m}(a_+ - a_-)^2 = -\frac{\hbar\omega}{4}(a_+a_+ - a_+a_- - a_-a_+ + a_-a_-)$. Evaluating $\langle T \rangle$, we have:

$$\langle T \rangle = \langle \psi_1 | T\psi_1 \rangle = -\frac{\hbar\omega}{4}\left[\langle \psi_1 | a_+a_+\psi_1 \rangle - \langle \psi_1 | a_+a_-\psi_1 \rangle - \langle \psi_1 | a_-a_+\psi_1 \rangle + \langle \psi_1 | a_-a_-\psi_1 \rangle\right]$$

$$\langle T \rangle = -\frac{\hbar\omega}{4}\left[\sqrt{2}\langle \psi_1 | a_+\psi_2 \rangle - \langle \psi_1 | a_+\psi_0 \rangle - \sqrt{2}\langle \psi_1 | a_-\psi_2 \rangle + \langle \psi_1 | a_-\psi_0 \rangle\right]$$

$$\langle T \rangle = -\frac{\hbar\omega}{4}\left[\sqrt{2}\sqrt{3}\langle \psi_1 | \psi_3 \rangle - \langle \psi_1 | \psi_1 \rangle - \sqrt{2}\sqrt{2}\langle \psi_1 | \psi_1 \rangle + \sqrt{0}\langle \psi_1 | \psi_{-1} \rangle\right] .$$

$$\langle T \rangle = -\frac{\hbar\omega}{4}[-1-2]$$

$$\langle T \rangle = \frac{3\hbar\omega}{4}$$

Space for calculation of $\langle V \rangle$.

Adding a_- and a_+ we can form an expression for x and we have

$x = \frac{\sqrt{2m\hbar\omega}}{2m\omega}(a_+ + a_-)$. To determine the expectation value of the potential energy, we will need to square the position operator and form the potential energy operator. Squaring we have:

$V = \frac{m\omega^2}{2}x^2 = \frac{2m\hbar\omega}{4m^2\omega^2}(a_+ + a_-)^2 = \frac{\hbar\omega}{4}(a_+a_+ + a_+a_- + a_-a_+ + a_-a_-)$. Evaluating $\langle V \rangle$, we have:

$$\langle V \rangle = \langle \psi_1 | V\psi_1 \rangle = \frac{\hbar\omega}{4}\left[\langle \psi_1 | a_+a_+\psi_1 \rangle + \langle \psi_1 | a_+a_-\psi_1 \rangle + \langle \psi_1 | a_-a_+\psi_1 \rangle + \langle \psi_1 | a_-a_-\psi_1 \rangle\right]$$

$$\langle V \rangle = \frac{\hbar\omega}{4}\left[\sqrt{2}\sqrt{3}\langle \psi_1 | \psi_3 \rangle + \langle \psi_1 | \psi_1 \rangle + \sqrt{2}\sqrt{2}\langle \psi_1 | \psi_1 \rangle + \sqrt{0}\langle \psi_1 | \psi_{-1} \rangle\right]$$

$$\langle V \rangle = \frac{\hbar\omega}{4}[1+2]$$

$$\langle V \rangle = \frac{3\hbar\omega}{4}$$

Notice that $\langle H \rangle = \frac{3\hbar\omega}{2}$ is simply the sum of $\langle T \rangle$ and $\langle V \rangle$, as it should be since $E_{n=1} = (n + \tfrac{1}{2})\hbar\omega = (1 + \tfrac{1}{2})\hbar\omega = \frac{3\hbar\omega}{2}$.

b. At what position(s) are you most and least likely to find the particle?

To evaluate the position(s) that you are most/least likely to find the particle we need the wave function. The first excited state wave function is:

$$|\psi_n\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega}{2\hbar}x^2}$$

$$|\psi_1\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^1 1!}} H_1\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega}{2\hbar}x^2}$$

We need to determine $H_1\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$. We can do this from

$H_n(q) = (-1)^n e^{q^2}\left(\frac{d}{dq}\right)^n e^{-q^2}$. Evaluating we find

$H_1(q) = (-1)^1 e^{q^2}\left(\frac{d}{dq}\right)^1 e^{-q^2} = -e^{q^2}\left(-2qe^{-q^2}\right) = 2q = \sqrt{\frac{2m\omega}{\hbar}}x$. Thus

$|\psi_1\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \sqrt{\frac{m\omega}{\hbar}} x e^{-\frac{m\omega}{2\hbar}x^2}$. The most/least likely location(s) are given by

$$\frac{d}{dx}\left(\psi_1^* \psi_1\right) = 0 \rightarrow \frac{d}{dx}\left(x^2 e^{-\frac{m\omega}{\hbar}x^2}\right) = 2xe^{-\frac{m\omega}{\hbar}x^2} - \frac{2m\omega}{\hbar}xx^2 e^{-\frac{m\omega}{\hbar}x^2} = x\left(1 - \frac{m\omega}{\hbar}x^2\right) = 0$$

$$x = 0;\; 1 = \frac{m\omega}{\hbar}x^2 \rightarrow x = \pm\sqrt{\frac{\hbar}{m\omega}}$$

Thus the most likely places are at $x = \pm\sqrt{\frac{\hbar}{m\omega}}$ and least likely is at $x = 0$.

I took the derivative and solved this by hand. If you used Mathematica, the code is below.

```
Solve[D[x^2*Exp[-(m*omega/hbar)*x^2], x] == 0, x]

Out[16]= {{x -> 0}, {x -> -Sqrt[hbar]/(Sqrt[m] Sqrt[omega])}, {x -> Sqrt[hbar]/(Sqrt[m] Sqrt[omega])}}
```

c. Suppose that the particle is in the ground state of the harmonic oscillator potential with frequency ω. Suppose that the frequency of oscillation suddenly doubles so that $\omega' = 2\omega$ without initially changing the wave function, what are the new energy states associated with the particle?

$$E_{n,\omega} = \left(n+\tfrac{1}{2}\right)\hbar\omega;\ n = 0,1,2,\ldots \rightarrow E_{n,\omega} = \frac{\hbar\omega}{2}, \frac{3\hbar\omega}{2}, \frac{5\hbar\omega}{2}, \ldots$$

$$E_{n,2\omega} = \left(n+\tfrac{1}{2}\right)\hbar(2\omega) = (2n+1)\hbar\omega;\ n = 0,1,2,\ldots \rightarrow E_{n,2\omega} = \hbar\omega, 3\hbar\omega, 5\hbar\omega, \ldots$$

d. For this particle in the first excited state of the harmonic oscillator potential with frequency ω, when the frequency suddenly doubles, what is the probability that a measurement of the energy would return $\hbar\omega$? Hint: In this case the probability is given by $\langle\psi|\psi'\rangle$, where $|\psi'\rangle$ is the new wave function with frequency 2ω and $|\psi\rangle$ is the original wave function with frequency ω.

$$|\psi_1\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \sqrt{\frac{m\omega}{\hbar}}\, x e^{-\frac{m\omega}{2\hbar}x^2} \text{ and}$$

$$|\psi_1'\rangle = \left(\frac{2m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \sqrt{\frac{2m\omega}{\hbar}}\, x e^{-\frac{2m\omega}{2\hbar}x^2} = 2^{\frac{3}{4}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \sqrt{\frac{m\omega}{\hbar}}\, x e^{-\frac{2m\omega}{2\hbar}x^2} = 2^{\frac{3}{4}} e^{-\frac{m\omega}{2\hbar}x^2} |\psi_1\rangle.$$

The probability is

$$P = \langle\psi_1'|\psi_1\rangle = 2^{\frac{3}{4}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{m\omega}{\hbar} \int_{-\infty}^{\infty} x^2 e^{-\frac{3m\omega}{2\hbar}x^2}\, dx = 2^{\frac{3}{4}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{m\omega}{\hbar}\left[\sqrt{\frac{8\pi\hbar^3}{4\times 27 m^3\omega^3}}\right].$$

$$P = 2^{\frac{3}{4}} \sqrt{\frac{m^3\omega^3}{\pi\hbar^3}\frac{2\pi\hbar^3}{27m^3\omega^3}} = 2^{\frac{5}{4}}\sqrt{\frac{1}{27}} = 0.458 = 45.8\%$$

I did the integral by hand. If you used Mathematica, the code for the integral and its evaluation is below.

```
coef = (2^(3/4)) * Sqrt[(m * omega) / (Pi * hbar)] * (m (omega / hbar))
Integrate[coef * x^2 * Exp[-((3 * m * omega) / (2 * hbar)) * x^2], {x, -Infinity, Infinity}]

Out[13]= (2^(3/4) m omega Sqrt[m omega / hbar]) / (hbar Sqrt[π])

Out[14]= ConditionalExpression[(2 × 2^(1/4)) / (3 Sqrt[3]), Re[m omega / hbar] > 0]

In[15]:= N[(2 × 2^(1/4)) / (3 Sqrt[3])]

Out[15]= 0.457726
```

2. Suppose that you have an electron of charge $-e$ and mass m in the state $|\psi_{321}\rangle$, where

$$|\psi_{321}\rangle = \frac{4}{81\sqrt{30}} a^{-\frac{3}{2}} \left(\tfrac{r}{a}\right)^2 e^{-\frac{r}{3a}} Y_l^m(\theta,\phi).$$

a. What is the most probable value of the radial coordinate?

The most probable value is where the derivative of the radial probability density function vanishes. The probability density function is given by

$$\frac{dP}{dr} = r^2 |R_{32}|^2 = r^2 \left[\frac{4}{81\sqrt{30}} a^{-\frac{3}{2}} \left(\tfrac{r}{a}\right)^2 e^{-\frac{r}{3a}} \right]^2 = \left(\frac{16}{196830 a^7} \right) r^6 e^{-\frac{2r}{3a}}$$

Taking the derivative and setting the result equal to zero gives the most probable value of the radial coordinate. We have

$$\frac{d}{dr}\left(\frac{dP}{dr}\right) = \frac{d}{dr}\left[\left(\frac{16}{196830 a^7} \right) r^6 e^{-\frac{2r}{3a}} \right] = 0$$

$$6r^5 e^{-\frac{2r}{3a}} - \frac{2r^6}{3a} e^{-\frac{2r}{3a}} = 0$$

$$r = 9a$$

I took the derivative by had. If you use Mathematica, the code is below.

```
Solve[D[(r^6 / a^7) * Exp[- (2 * r) / (3 * a)], r] == 0, r]
Out[12]= {{r → 0}, {r → 0}, {r → 0}, {r → 0}, {r → 0}, {r → 9 a}}
```

b. What is the probability of finding the electron between the nucleus and the four times the Bohr radius, $4a$? Hints: You may need the following integrals:

$$\int \sin^n\theta\, d\theta = -\frac{\sin^{n-1}\theta\cos\theta}{n} + \frac{n-1}{n}\int \sin^{n-2}\theta\, d\theta;\quad n \ge 0$$

$$\int \cos^n\theta\, d\theta = -\frac{\cos^{n-1}\theta\sin\theta}{n} + \frac{n-1}{n}\int \cos^{n-2}\theta\, d\theta;\quad n \ge 0.$$

$$\int x^n e^{ax} = \frac{e^{ax}}{a}\left(x^n - \frac{nx^{n-1}}{a} + \frac{n(n-1)x^{n-2}}{a^2} - \ldots \frac{(-1)^n n!}{a^n}\right)$$

There are two ways to do this problem. This is the long version. To determine the probability, integrate the wave function over all space. To do this we need to know what the wave function looks like and thus we need to evaluate the spherical harmonic corresponding to this state.

$$Y_l^{m_l}(\theta,\phi) = \varepsilon\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m_l|)!}{(l+|m_l|)!}}e^{im\phi}P_l^{m_l}(\cos\theta);\quad \varepsilon = \begin{cases} (-1)^{m_l} & m_l \ge 0 \\ 1 & m_l \le 0 \end{cases}$$

$$P_l^{m_l}(\cos\theta) = \left(1-\cos^2\theta\right)^{\frac{|m_l|}{2}}\left(\frac{d}{d\cos\theta}\right)^{|m_l|} P_l(\cos\theta)$$

$$P_l(\cos\theta) = \frac{1}{2^l l!}\left(\frac{d}{d\cos\theta}\right)^l \left(\cos^2\theta - 1\right)^l$$

Thus,

$$Y_2^1 = (-1)^1\sqrt{\frac{(5)}{4\pi}\frac{(1)!}{(3)!}}e^{i\phi}P_1^1(\cos\theta) = -\sqrt{\frac{5}{24\pi}}\left[\left(1-\cos^2\theta\right)^{\frac{1}{2}}\left(\frac{d}{d\cos\theta}\right)^1 P_2(\cos\theta)\right]e^{i\phi}$$

$$Y_2^1 = -\sqrt{\frac{5}{24\pi}}\sin\theta\left[\frac{d}{dx}\left(\frac{1}{2^2 2!}\frac{d}{dx}\left(\frac{d}{dx}\left(x^2-1\right)^2\right)\right)\right]e^{i\phi}$$

Where we have defined $x = \cos\theta$. Performing the derivatives and constructing the spherical harmonic we have

$Y_2^1 = -\sqrt{\frac{5}{24\pi}}\sin\theta\left[3\cos\theta\right]e^{i\phi}$ where we have converted the result back to a function of theta.

The simplified wave function for this state is

$$|\psi_{321}\rangle = -\frac{1}{81\sqrt{\pi}}a^{-\frac{7}{2}}r^2 e^{-\frac{r}{3a}}\sin\theta\cos\theta e^{i\phi}$$

The probability is $P = \langle\psi_{321}|\psi_{321}\rangle$.

Evaluating:

$$P=\langle\psi_{321}|\psi_{321}\rangle=\left(\frac{1}{81\sqrt{\pi}}a^{-\frac{7}{2}}\right)^2\int_0^{4a}\left(r^2e^{-\frac{r}{3a}}\right)^2 r^2\,dr\int_0^{\pi}(\sin\theta\cos\theta)^2\,sin\theta\,d\theta\int_0^{2\pi}d\phi$$

$$P=\frac{2\pi}{6561\pi a^7}\left[\int_0^{4a}r^6e^{-\frac{2r}{3a}}\,dr\right]\left[\int_0^{\pi}sin^3\theta\,d\theta-\int_0^{\pi}sin^5\theta\,d\theta\right]$$

$$P=\frac{2}{6561a^7}\left[\frac{3}{8}a^7\left(32805-\frac{462973}{e^{\frac{8}{3}}}\right)\right]\left[\frac{4}{3}-\frac{16}{15}\right]=\frac{15265}{787320}$$

$$P=0.019=1.9\%$$

The radial integral was evaluated on Mathematica and the theta and phi integrals were evaluated by hand. The mathematica code for the radial (and theta if you did it on Mathematica) integrals are shown below.

```
In[5]:= Integrate[r^6*Exp[-(2*r)/(3*a)], {r, 0, 4*a}]

Out[5]= 3/8 a^7 (32 805 - 462 973/e^(8/3))

In[6]:= N[32 805 - 462 973/e^(8/3)]

Out[6]= 636.038
```

```
In[7]:= Integrate[Sin[t]^3*Cos[t]^2, {t, 0, Pi}]

Out[7]= 4/15
```

The short version of doing this problem, doesn't involve evaluating the spherical harmonic at all. We have

$$P=\langle\psi_{321}|\psi_{321}\rangle=\left(\frac{4}{81\sqrt{30a^3}a^2}\right)^2\int_0^{4a}\left(r^2e^{-\frac{r}{3a}}\right)^2 r^2\,dr\left[\int_0^{\pi}\int_0^{2\pi}Y_2^1\sin\theta\,d\theta\,d\phi\right]$$

$$P=\frac{8.13\times10^{-5}}{a^7}\left[\int_0^{4a}r^6e^{-\frac{2r}{3a}}\,dr\right][1]$$

$$P=\frac{8.13\times10^{-5}}{a^7}\left[\frac{3}{8}a^7\left(32805-\frac{462973}{e^{\frac{8}{3}}}\right)\right]=\left(\frac{8.13\times10^{-5}}{a^7}\right)\times\left(238.5a^7\right)$$

$$P=0.019=1.9\%$$

And the radial integral was done directly above on Mathematica and we used the fact that the spherical harmonics are normalized.

c. Is the state $|\psi_{321}\rangle$ an eigenstate of the z-component of angular momentum? If it is, what is $\langle L_z\rangle$? If it is not, explain why it is not.

$$L_Z|\psi_{321}\rangle = -i\hbar\frac{d}{d\phi}\left(-3\sqrt{\frac{5}{24\pi}}\frac{4}{81\sqrt{30}}a^{-\frac{3}{2}}\left(\tfrac{r}{a}\right)^2 e^{-\frac{r}{3a}}\sin\theta\cos\theta e^{i\phi}\right) = -i\hbar(i)|\psi_{321}\rangle = \hbar|\psi_{321}\rangle$$

Thus the state $|\psi_{321}\rangle$ an eigenstate of the z-component of angular momentum with eigenvalue $\hbar$. The expectation value of the z-component of the angular momentum is $\langle L_z\rangle = \langle\psi_{321}|L_z\psi_{321}\rangle = \hbar\langle\psi_{321}|\psi_{321}\rangle = \hbar$.

d. Suppose that the electron transitions form the state $|\psi_{321}\rangle$ to the state $|\psi_{100}\rangle$. In this process a photon is emitted with energy ΔE. What is the energy of the emitted photon in eV?

$$\Delta E = E_{upper} - E_{lower} = -\frac{13.6eV}{n_{upper}^2} - \left(-\frac{13.6eV}{n_{lower}^2}\right) = \frac{13.6eV}{n_{lower}^2} - \frac{13.6eV}{n_{upper}^2}$$

$$\Delta E = 13.6eV\left(\frac{1}{1^2} - \frac{1}{3^2}\right) = 12.1eV$$

Physics 220 Equations

Useful Integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int \cos^2(qx)dx = \frac{x}{2} + \frac{\sin[2qx]}{4q}$$

$$\int \sin^2(qx)dx = \frac{x}{2} - \frac{\sin[2qx]}{4q}$$

$$\int \cos^3(qx)dx = \frac{3\sin[qx]}{4q} + \frac{\sin[3qx]}{12q}$$

$$\int \sin^3(qx)dx = -\frac{3\cos[qx]}{4q} + \frac{\cos[3qx]}{12q}$$

$$\int_{-a/2}^{a/2} x\cos^2(qx)dx = 0$$

$$\int_{-a/2}^{a/2} x\sin^2(qx)dx = 0$$

$$\int_{-a/2}^{a/2} \sin(qx)\cos(qx)dx = 0$$

$$\int e^{\pm ax}\,dx = \pm\frac{e^{\pm ax}}{a}$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{\infty} xe^{-ax^2}\,dx = 0$$

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{2a^{\frac{3}{2}}}$$

$$2\alpha\sqrt{\tfrac{\alpha}{\pi}}\int x^2 e^{-\alpha x^2}\,dx = -\sqrt{\tfrac{\alpha}{\pi}}\,xe^{-\alpha x^2} + \frac{\mathrm{Erf}[\sqrt{\alpha}x]}{2};\quad \mathrm{Erf}[0]\equiv 0$$

$$\int_{400nm}^{700nm} \frac{2\pi hc^2}{\lambda^5}\left[\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right]d\lambda = 1197\,\tfrac{W}{m^2}$$

Constants:

$g = 9.8\,\frac{m}{s^2}$

$G = 6.67\times10^{-11}\,\frac{Nm^2}{kg^2}$

$c = 3\times10^{8}\,\frac{m}{s}$

$\sigma = 5.67\times10^{-8}$

$k_B = 1.38\times10^{-23}\,\frac{J}{K}$

$1eV = 1.6\times10^{-19}\,J$

$1e = 1.6\times10^{-19}\,C$

$h = 6.63\times10^{-34}\,Js;$

$m_e = 9.11\times10^{-31}\,kg = 0.511\,\frac{MeV}{c^2}$

$m_p = 1.67\times10^{-27}\,kg = 938\,\frac{MeV}{c^2}$

$m_n = 1.69\times10^{-27}\,kg = 939\,\frac{MeV}{c^2}$

$m_E = 6\times10^{24}\,kg$

$R_E = 6.4\times10^{6}\,m$

$a = 0.5\times10^{-10}\,m$

Formulas :

$c = \upsilon\lambda$

$E = h\upsilon = \dfrac{hc}{\lambda}$

$$\frac{dS}{d\lambda} = \frac{2\pi hc^2}{\lambda^5}\left[\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right]$$

$$\frac{dS}{d\upsilon} = \frac{2\pi h\upsilon^3}{c^2}\left(\frac{1}{e^{\frac{h\upsilon}{kT}}-1}\right)$$

$$\frac{dS}{d\lambda} = \frac{2\pi ckT}{\lambda^4}$$

$$\lambda_{\max} = \frac{2.9\times10^{-3}\,m\cdot K}{T}$$

$S = \sigma T^4$

$eV_{stop} = hf - \phi$

$$\lambda' = \lambda + \frac{h}{mc}(1-\cos\phi)$$

$$\hbar = \frac{h}{2\pi};\; k = \frac{2\pi}{\lambda};\; \omega = 2\pi f$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t} = E\psi$$

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

$$\hat{p} = -i\hbar\frac{\partial}{dx}$$

$$\hat{T} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V$$

$\hat{x} = x$

$$\langle O\rangle = \int \psi^*\hat{O}\psi\,dr$$

$$P = \int \psi^*\psi\,dx$$

$$E_n = n^2\left(\frac{\pi^2\hbar^2}{2ma^2}\right)$$

$$\Psi_n(x,t) = \sqrt{\frac{2}{a}}\sin(k_n x)\,e^{-i\frac{E_n}{\hbar}t}$$

$$T = \frac{k'}{k}\left|\frac{F}{A}\right|^2$$

$$T + R = 1$$

$$T = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)}\sinh^2\left(\sqrt{\frac{8ma^2}{\hbar^2}(V_0 - E)}\right)}; \;\; E < V_0$$

$$T = \frac{1}{1 + a^2k^2}; \;\; E \sim V_0$$

$$T = \frac{1}{1 + \frac{V_0^2}{4E(E - V_0)}\sin^2\left(\sqrt{\frac{8ma^2}{\hbar^2}(E - V_0)}\right)}; \;\; E > V_0$$

$$H_n(q) = (-1)^n e^{q^2}\left(\frac{d}{dq}\right)^n e^{-q^2}; \;\; q = \sqrt{\tfrac{m\omega}{2\hbar}}x$$

$$Y_l^{m_l}(\theta,\phi) = \varepsilon\sqrt{\frac{(2l+1)}{4\pi}\frac{\left(l - |m_l|\right)!}{\left(l + |m_l|\right)!}}e^{im\phi}P_l^{m_l}(\cos\theta); \quad \varepsilon = \begin{cases} (-1)^{m_l} & m_l \ge 0 \\ 1 & m_l \le 0 \end{cases}$$

$$P_l^{m_l}(\cos\theta) = \left(1 - \cos^2\theta\right)^{\frac{|m_l|}{2}}\left(\frac{d}{d\cos\theta}\right)^{|m_l|}P_l(\cos\theta)$$

$$P_l(\cos\theta) = \frac{1}{2^l l!}\left(\frac{d}{d\cos\theta}\right)^l\left(\cos^2\theta - 1\right)^l$$

$$L_{n-l-1}^{2l+1}\left(\tfrac{2r}{na}\right) = (-1)^{2l+1}\left(\tfrac{na}{2}\right)^{2l+1}\left(\tfrac{d}{dr}\right)^{2l+1}L_{n+2l}\left(\tfrac{2r}{na}\right)$$

$$L_{n+2l}\left(\tfrac{2r}{na}\right) = e^{\frac{2r}{na}}\left(\tfrac{na}{2}\right)^{n+2l}\left(\tfrac{d}{dr}\right)^{n+2l}\left(e^{-\frac{2r}{na}}\left(\tfrac{2r}{na}\right)^{n+2l}\right)$$

$$\left|\psi_{nlm_l}\right\rangle = \sqrt{\left(\tfrac{2}{na}\right)^3\frac{(n-l-a)!}{2n\left[(n+l)!\right]^3}}e^{-\frac{2r}{na}}\left(\tfrac{2r}{na}\right)^l\left[L_{n-l-1}^{2l+1}\left(\tfrac{2r}{na}\right)\right]Y_l^{m_l}(\theta,\phi)$$

$$a_\pm = \frac{1}{\sqrt{2m\hbar\omega}}\left(\mp ip + m\omega x\right)$$

$$H = \left(a_\pm a_\mp \pm \tfrac{1}{2}\right)\hbar\omega$$

$$L_\pm = L_x \pm iL_y$$

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$P = \int\psi^*\psi\, d^3r = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\psi^*\psi\, r^2\, dr\sin\theta\, d\theta d\phi$$