Physics 220
Homework #1
Spring 2017
Due Wednesday, 4/5/17

1. Consider the function $y = Axe^{-\frac{x}{2a}}$ over the region $0 \le x \le \infty$. Perform the following integral on Mathemtatica ($\int_0^\infty y^2 x^2\, dx$) and set the result equal to one and determine the constant A.

   The solution using Mathematica is given below.

```
(*  Homework #1, Problem #1  *)
(*  We first define the function y as given in the problem.  Then we integrate
 the function over the range of zero to infinity.  To determine the unknown
 coefficient we set the result equal to one and use the solve command to
 determine A and take the positive solution. *)

In[4]:= y = A*x*Exp[-x/(2*a)]
```

Out[4]= $A\, e^{-\frac{x}{2a}}\, x$

```
In[5]:= Integrate[y^2*x^2, {x, 0, Infinity}]
```

Out[5]= ConditionalExpression[$24\, a^5 A^2$, Re[a] > 0]

```
In[6]:= Solve[24 a^5 A^2 == 1, A]
```

Out[6]= $\left\{\left\{A \to -\frac{1}{2\sqrt{6}\, a^{5/2}}\right\}, \left\{A \to \frac{1}{2\sqrt{6}\, a^{5/2}}\right\}\right\}$

(* Thus the solution is: $A = \frac{1}{2\sqrt{6}\, a^{5/2}}$. *)

2. Consider the following functions $y = 4x - 2$ and $y = x^2 + 2x - 3$ over the region $-5 \le x \le 5$. Using Mathematica, plot both of these functions on the same graph over the region of $-5 \le x \le 5$. Label the axes and title the plot. Using the *FindRoot* command, what are the points of intersection? Lastly, using the *Solve* command, what are the points of intersection? How do your two results compare?

The solution using Mathematica is given below.

```
(*  Homework #1, Problem #2  *)
(*  We first plot the functions y as are given in the problem.  Then we use
 the find root command to determine intersection points.  Lastly we compare
 the results with the solve command and see that they are identical. *)

In[10]:= Plot[{4*x-2, x^2+2*x-3}, {x, -5, 5}, AxesLabel → {"x", "y"}]

Out[10]=
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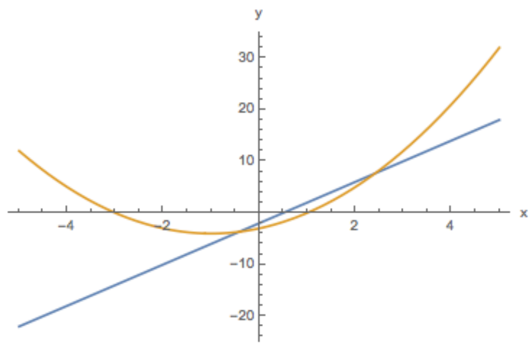



```
In[11]:= FindRoot[4*x-2 == x^2+2*x-3, {x, 2.5}]
         FindRoot[4*x-2 == x^2+2*x-3, {x, -0.5}]

Out[11]= {x → 2.41421}

Out[12]= {x → -0.414214}

In[13]:= Solve[4*x-2 == x^2+2*x-3, x]

Out[13]= {{x → 1 - √2}, {x → 1 + √2}}
```

3. A particle of mass m is moving in one dimension in a potential $V(x,t)$. The wave function for the particle is $\Psi(x,t) = Axe^{-\left(\frac{\sqrt{km}}{2\hbar}\right)x^2} e^{-i\left(\frac{3}{2}\sqrt{\frac{k}{m}}\right)t}$ for $-\infty < x < \infty$, where A and k are constants.

a. Normalize the wave function and determine the constant A.
Using the normalization condition, we have

$$\int \Psi^* \Psi dx = 1 \rightarrow 1 = \int_{-\infty}^{\infty} A^2 x^2 e^{-\frac{2\sqrt{km}}{2\hbar}x^2} dx = A^2 \left(\frac{\sqrt{\pi}}{2\left(\frac{\sqrt{km}}{\hbar}\right)^{\frac{3}{2}}} \right)$$

$$\therefore A = \frac{\sqrt{2}}{\pi^{\frac{1}{4}}} \left(\frac{\sqrt{km}}{\hbar}\right)^{\frac{3}{4}}$$

The integral and subsequent solving for A were done on Mathematica. The code is below.

```
(*  Homework #1, Problem #3a  *)
(*  We integrate the given function over the limits minus infinity to plus
 infinity.  Then we copy the results of teh integration and use the solve
 command to solve for the normalization constant.  We then take the positive
 solution as our result.  *)

In[2]:= Integrate[(A^2*x^2)*Exp[-(Sqrt[k*m]/hbar)*x^2], {x, -Infinity, Infinity}]

Out[2]= ConditionalExpression[(A^2 Sqrt[π])/(2 (Sqrt[k m]/hbar)^(3/2)), Re[Sqrt[k m]/hbar] > 0]

In[3]:= Solve[(A^2 Sqrt[π])/(2 (Sqrt[k m]/hbar)^(3/2)) == 1, A]

Out[3]= {{A -> -(Sqrt[2] (Sqrt[k m]/hbar)^(3/4))/π^(1/4)}, {A -> (Sqrt[2] (Sqrt[k m]/hbar)^(3/4))/π^(1/4)}}
```

b. Using the normalized wave function, calculate $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, and $\langle p^2\rangle$.

$$\langle x\rangle = \int_{-\infty}^{\infty}\Psi^* x\Psi\,dx = A^2\int_{-\infty}^{\infty} x^3 e^{-\frac{2\sqrt{km}}{2\hbar}x^2}\,dx = 0$$

$$\langle p\rangle = \int_{-\infty}^{\infty}\Psi^*\left(-i\hbar\frac{d}{dx}\right)\Psi\,dx = A^2\left[\int_{-\infty}^{\infty} x e^{-\frac{2\sqrt{km}}{2\hbar}x^2}\,dx - \frac{2\sqrt{km}}{2\hbar}\int_{-\infty}^{\infty} x^3 e^{-\frac{2\sqrt{km}}{2\hbar}x^2}\,dx\right] = 0$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty}\Psi^* x^2\Psi\,dx = A^2\int_{-\infty}^{\infty} x^4 e^{-\frac{2\sqrt{km}}{2\hbar}x^2}\,dx = A^2\frac{3\sqrt{\pi}}{4}\left(\frac{\hbar}{\sqrt{km}}\right)^{\frac{5}{2}} = \frac{3\sqrt{km}}{20\hbar}$$

$$\langle p^2\rangle = \int_{-\infty}^{\infty}\Psi^*\left(-i\hbar\frac{d}{dx}\right)\left(-i\hbar\frac{d}{dx}\right)\Psi\,dx = \hbar^2\int_{-\infty}^{\infty}\Psi^*\frac{d^2\Psi}{dx^2}dx = \hbar^2\int_{-\infty}^{\infty}\Psi^*\frac{d^2\Psi}{dx^2}dx$$

$$=$$

c. Show that V is independent of t, and determine $V(x)$.

$$\Psi = Axe^{-\frac{\sqrt{km}}{2\hbar}x^2}e^{-i\frac{3}{2}\sqrt{\frac{k}{m}}t}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + V\Psi = E\Psi$$

$$E\Psi = i\hbar\frac{d\Psi}{dt} = i\hbar\frac{d}{dt}\left(Axe^{-\frac{\sqrt{km}}{2\hbar}x^2}e^{-i\frac{3}{2}\sqrt{\frac{k}{m}}t}\right) = \frac{3\hbar}{2}\sqrt{\frac{k}{m}}\Psi$$

$$\frac{d\Psi}{dx} = \frac{d}{dx}\left(Axe^{-\frac{\sqrt{km}}{2\hbar}x^2}e^{-i\frac{3}{2}\sqrt{\frac{k}{m}}t}\right) = Ae^{-\frac{\sqrt{km}}{2\hbar}x^2}e^{-i\frac{3}{2}\sqrt{\frac{k}{m}}t} + Ax\left(-\frac{2x\sqrt{km}}{2\hbar}e^{-\frac{\sqrt{km}}{2\hbar}x^2}e^{-i\frac{3}{2}\sqrt{\frac{k}{m}}t}\right)$$

$$\frac{d^2\Psi}{dx^2} = \frac{d}{dx}\left(\frac{d\Psi}{dx}\right) = -\frac{\sqrt{km}}{2\hbar}6\Psi + \frac{km}{4\hbar^2}4x^2\Psi$$

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + V\Psi = E\Psi \to -\frac{\hbar^2}{2m}\left(-\frac{\sqrt{km}}{2\hbar}6\Psi + \frac{km}{4\hbar^2}4x^2\Psi\right) + V\Psi = \frac{3\hbar}{2}\sqrt{\frac{k}{m}}\Psi$$

$$\therefore V = \hbar\left(\frac{3}{2}\sqrt{\frac{k}{m}} - \frac{1}{2}\sqrt{\frac{k}{m}} - \sqrt{\frac{k}{m}}\right) + \frac{1}{2}kx^2 = \frac{1}{2}kx^2$$

4. Determine which of the following one-dimensional wave functions represent state of definite momentum. For each wave function that does correspond to a state of definite momentum, determine the momentum.

a. $\psi(x) = e^{ikx}$

$$-i\hbar\frac{d\psi}{dx} = -i\hbar\frac{d}{dx}\left(e^{ikx}\right) = -i\hbar(ik)e^{ikx} = \hbar k\psi = p\psi \therefore p = \hbar k$$

b. $\psi(x) = xe^{ikx}$

$$-i\hbar\frac{d\psi}{dx} = -i\hbar\frac{d}{dx}\left(xe^{ikx}\right) = -i\hbar\left(e^{ikx} + ikxe^{ikx}\right) \neq p\psi \therefore NO$$

c. $\psi(x) = \sin(kx) + i\cos(kx)$

$$-i\hbar\frac{d\psi}{dx} = -i\hbar\frac{d}{dx}(\sin kx + i\cos kx) = -i\hbar(k\cos kx - ik\sin kx) = -\hbar k\psi \therefore p = -\hbar k$$

d. $\psi(x) = e^{ikx} + e^{-ikx}$

$$-i\hbar\frac{d\psi}{dx} = -i\hbar\frac{d}{dx}\left(e^{ikx} + e^{-ikx}\right) = -i\hbar\left(ike^{ikx} - ike^{-ikx}\right) = \hbar k\left(e^{ikx} - e^{-ikx}\right) \neq p\psi \therefore NO$$

5. Griffith's problem 1.4.

a. Normalizing the wave function we have:

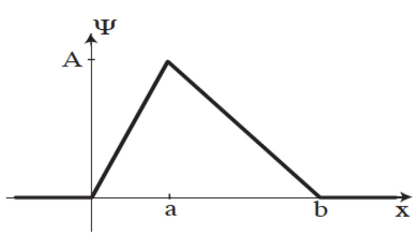
$$1 = \int \psi^*\psi\, dx = \int_0^a \left(\frac{Ax}{a}\right)^2 dx + \int_a^b \left(\frac{A(b-x)}{b-a}\right)^2 dx$$

$$1 = \frac{A^2}{a^2}\int_0^a x^2\, dx + \frac{A^2}{(b-a)^2}\int_a^b (b-x)^2\, dx$$

$$1 = \frac{A^2}{a^2}\left(\frac{a^3}{3}\right) - \frac{A^2}{(b-a)^2}\left(0 - \frac{(b-a)^3}{3}\right) = A^2\left(\frac{a}{3} + \frac{b-a}{3}\right) = A^2\frac{b}{3}$$

$$\therefore A = \sqrt{\frac{3}{b}}$$

b.



c. From the graph the most probably location is at $x = a$.

d. To calculate the probabilities:

$$P = \int \psi^* \psi \, dx = \int_0^a \left(\frac{Ax}{a} \right)^2 dx = \frac{A^2}{a^2}\left(\frac{a^3}{3} \right) = \frac{3a^3}{3ba^2} = \frac{a}{b}.$$

We check the case that $a = b$, and we see that the probability is Unity. We check the case that $2a = b$ an isosceles triangle, and we see that the probability is $P = \frac{a}{b} = \frac{a}{2a} = \frac{1}{2}$.

e. To calculate the expectation value of the position, we evaluate:

$$\langle x \rangle = \int \psi^* x \psi \, dx = A^2 \left\{ \frac{1}{a^2} \int_0^a x^3 \, dx + \frac{1}{(b-a)^2} \int_a^b x(b-x)^2 \, dx \right\}$$

$$\langle x \rangle = \frac{3}{b} \left\{ \frac{1}{a^2} \left(\frac{x^2}{4} \right) \Bigg|_0^a + \frac{1}{(b-a)^2} \left(b^2 \frac{x^2}{2} - 2b \frac{x^3}{3} + \frac{x^4}{4} \right) \Bigg|_a^b \right\}$$

$$\langle x \rangle = \frac{1}{4(b-a)^2} \left(\frac{b^4}{3} - 3a^2 b + 2a^3 \right) = \frac{2a+b}{4}$$

This was also done on Mathematica and the code is shown below.

```
(*  Homework #1, Problem #5e  *)

In[14]:= Clear[A, a, b, x, y]

In[15]:= Integrate[(A^2 / a^2) * x^3, {x, 0, a}]

Out[15]= a^2 A^2 / 4

In[17]:= Integrate[(A^2 / (b - a) ^2) * x * (b - x) ^2, {x, a, b}]

Out[17]= -1/12 A^2 (a - b) (3 a + b)

In[18]:= A = Sqrt[3 / b]

Out[18]= Sqrt[3] Sqrt[1/b]

In[19]:= Simplify[a^2 A^2 / 4 - 1/12 A^2 (a - b) (3 a + b)]

Out[19]= 1/4 (2 a + b)
```

6. Griffith's problem 1.5.

a. We use the normalization condition $P = \int \Psi^* \Psi dx = 1$ to determine the coefficient A. On Mathematica we have:

```
(*  Homework #1, Problem #6  *)

(*  Part a, normalize the wavefunction  *)

In[21]:= Clear[A, a, b, x]

In[23]:= Integrate[A^2*Exp[-2*l*x], {x, 0, Infinity}]

Out[23]= ConditionalExpression[A^2/(2 l), Re[l] > 0]

(*  To do the integral we only used half of the range and then multiplied by
  two.  If you do the whole range the intergral does not converge to a finite
  value because of the discontinuity at x = 0  *)

In[24]:= Solve[2*A^2/(2 l) == 1, A]

Out[24]= {{A -> -Sqrt[l]}, {A -> Sqrt[l]}}
```

b. To calculate the expectation values of the $\langle x \rangle$ and $\langle x^2 \rangle$: we compute $\langle x \rangle = \int \Psi^* x \Psi dx$ and $\langle x^2 \rangle = \int \Psi^* x^2 \Psi dx$ on Mathematica.

```
(*  Homework #1, Problem #6  *)

(*  Part b, expectation values  *)

In[30]:= Integrate[A^2*x*Exp[-2*l*x], {x, 0, Infinity}]

Out[30]= ConditionalExpression[A^2/(4 l^2), Re[l] > 0]

In[28]:= Integrate[A^2*x*Exp[-2*l*x], {x, -Infinity, 0}]

Out[28]= ConditionalExpression[-A^2/(4 l^2), Re[l] < 0]

(*  The expectation value of the position <x>
  is the sum of these two results with is of course zero.  *)

In[34]:= Integrate[A^2*x^2*Exp[-2*l*x], {x, 0, Infinity}]

Out[34]= ConditionalExpression[A^2/(4 l^3), Re[l] > 0]

(*  The expectation value of the position <x^2>
  is twice this expression since we try to avoid the discontinuity at x =
  0.  Thus the answer is 2(A^2/(4 l^3)) = 2(1/(4 l^3)) = 1/(2 l^2)  *)
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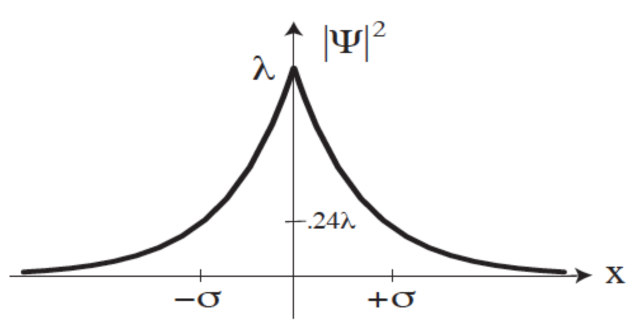
And just to show you what this looks like, we plot the wave function. Coincidentally this wave function corresponds to a Delta function potential. We'll see this in the course of our studv of wells and barriers.



7. Suppose that ψ_1 and ψ_2 are two different solutions of the time-independent Schrodinger wave equation with the same energy E.

a. Show that $\psi_1 + \psi_2$ is also a solution with energy E.

$$\psi_1 : -\frac{\hbar^2}{2m}\nabla^2\psi_1 + V\psi_1 = E\psi_1$$

$$\psi_2 : -\frac{\hbar^2}{2m}\nabla^2\psi_2 + V\psi_2 = E\psi_2$$

$$Add: \ -\frac{\hbar^2}{2m}\nabla^2\left(\psi_1 + \psi_2\right) + V\left(\psi_1 + \psi_2\right) = E\left(\psi_1 + \psi_2\right)$$

$$\therefore -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi; \ \ \psi = \psi_1 + \psi_2$$

Thus since ψ satisfies the SWE and is a solution, the sum $\psi_1 + \psi_2$ must also satisfy the SWE.

b. Show that $c\psi_1$ is also a solution of the Schrodinger equation with energy E.

$$\psi_1 : -\frac{\hbar^2}{2m}\nabla^2\psi_1 + V\psi_1 = E\psi_1$$

$$\psi_2 = c\psi_1 : -\frac{\hbar^2}{2m}\nabla^2\psi_2 + V\psi_2 = E\psi_2 \rightarrow -\frac{\hbar^2}{2m}c\nabla^2\psi_1 + Vc\psi_1 = Ec\psi_1$$

$$\therefore -\frac{\hbar^2}{2m}\nabla^2\psi_1 + V\psi_1 = E\psi_1$$

which is a solution to the wave equation, so $c\psi_1$ is a solution to the wave equation.

8. A particle of mass m is moving in one dimension near the speed of light so that the relation for the kinetic energy $E = \frac{p^2}{2m}$ is no longer valid. Instead, the total energy is given by $E^2 = p^2c^2 + m^2c^4$. So, we can no longer use the Schrodinger equation. Suppose that the wave function for the particle $\Psi(x,t)$ is an eigenfunction of the energy operator ($i\hbar\frac{d}{dt}|\Psi\rangle = E|\Psi\rangle$) and an eigenfunction of the momentum operator ($-i\hbar\frac{d}{dx}|\Psi\rangle = p|\Psi\rangle$). If there is no potential energy V, derive a linear differential equation for $\Psi(x,t)$.

We start with the relativistic energy equation and then use the definitions of the energy and momentum operators.

$$E^2 = p^2c^2 + m^2c^4$$

$$i\hbar\frac{d\psi}{dt} = E\psi \rightarrow i\hbar\frac{d}{dt}\left(i\hbar\frac{d\psi}{dt}\right) = i\hbar\frac{d}{dt}(E\psi) = E\left(i\hbar\frac{d\psi}{dt}\right) = E^2\psi$$

$$\therefore -\hbar^2\frac{d^2\psi}{dt^2} = E^2\psi$$

$$-i\hbar\frac{d\psi}{dx} = p\psi \rightarrow -i\hbar\frac{d}{dx}\left(-i\hbar\frac{d\psi}{dx}\right) = -i\hbar\frac{d}{dx}(p\psi) = p^2\psi$$

$$\therefore -\hbar^2\frac{d^2\psi}{dx^2} = p^2\psi$$

$$SWE:\ E^2\psi = p^2c^2\psi + m^2c^4\psi \rightarrow -\hbar^2\frac{d^2\psi}{dt^2} = -\hbar^2c^2\frac{d^2\psi}{dx^2} + m^2c^4\psi$$

$$\therefore \frac{1}{c^2}\frac{d^2\psi}{dt^2} = \frac{d^2\psi}{dx^2} - \left(\frac{mc}{\hbar}\right)^2\psi$$

This equation is called the Klein-Gordon equation and it forms part of the basis for the study of relativistic quantum mechanics. It's the relativistic form of the Schrodinger wave equation.

9. Griffith's Problem 1.15

a. To show this result, we start with the definition of probability.

$P=\int_{-\infty}^{\infty}\Psi^*\Psi\,dx=\int_{-\infty}^{\infty}|\Psi|^2\,dx$. Now if is normalized does it stay normalized and what happens if the potential function is not real but contains an imaginary part? So, to see if it stay normalized we take the time derivative. If the time derivative is zero then the probability function will stay normalized. Let's take the time derivative (and note that the integral and time derivative commute):

$$\frac{dP}{dt}=\frac{d}{dt}\left(\int_{-\infty}^{\infty}|\Psi|^2\,dx\right)=\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left(|\Psi|^2\right)dx=\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left(\Psi^*\Psi\right)dx=\int_{-\infty}^{\infty}\left[\Psi^*\frac{\partial\Psi}{\partial t}+\frac{\partial\Psi^*}{\partial t}\Psi\right]dx.$$

Now we need expressions for the terms in the brackets.

$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V\Psi=i\hbar\frac{\partial\Psi}{\partial t}\rightarrow\frac{\partial\Psi}{\partial t}=\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2}-\frac{iV}{\hbar}\Psi$ and the complex conjugate

$\frac{\partial\Psi^*}{\partial t}=-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}+\frac{iV^*}{\hbar}\Psi^*$, assuming that the potential can be complex. Back to the expression in brackets.

$$\left[\Psi^*\frac{\partial\Psi}{\partial t}+\frac{\partial\Psi^*}{\partial t}\Psi\right]=\Psi^*\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2}-\frac{iV}{\hbar}\Psi\right)+\left(-\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2}+\frac{iV^*}{\hbar}\Psi^*\right)\Psi$$

$$=\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2}-\frac{\partial^2\Psi^*}{\partial x^2}\Psi\right)+\frac{i}{\hbar}\left(-V\Psi^*\Psi+V^*\Psi^*\Psi\right)$$

If the potential function is given as $V=V_0+i\Gamma$ and taking the complex conjugate $V^*=V_0-i\Gamma$ we can evaluate the last term on the right and we find

$\frac{i}{\hbar}\left(-V+V^*\right)\Psi^*\Psi=\frac{i}{\hbar}\left(V_0-i\Gamma-V_0+i\Gamma\right)\Psi^*\Psi=-\frac{2\Gamma}{\hbar}\Psi^*\Psi=-\frac{2\Gamma}{\hbar}|\Psi|^2$. Now finishing the problem.

$$\frac{dP}{dt}=\int_{-\infty}^{\infty}\left[\Psi^*\frac{\partial\Psi}{\partial t}+\frac{\partial\Psi^*}{\partial t}\Psi\right]dx=\int_{-\infty}^{\infty}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2}-\frac{\partial^2\Psi^*}{\partial x^2}\Psi\right)-\frac{2\Gamma}{\hbar}|\Psi|^2\right]dx$$

$$\frac{dP}{dt}=\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)\right]-\frac{2\Gamma}{\hbar}\int_{-\infty}^{\infty}|\Psi|^2\,dx=\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{\infty}-\frac{2\Gamma}{\hbar}P$$

$$\frac{dP}{dt}=-\frac{2\Gamma}{\hbar}P$$

where evaluating the expression in parenthesis is zero since the wave function is normalized it has to vanish at $\pm\infty$.

b. From part a, we can solve for the probability as a function of time. We have:

$$\frac{dP}{dt} = -\frac{2\Gamma}{\hbar}P \rightarrow \frac{dP}{P} = -\frac{2\Gamma}{\hbar}dt \rightarrow \ln P = -\frac{2\Gamma}{\hbar}t$$
$$\therefore P(t) = P_0 e^{-\frac{2\Gamma}{\hbar}t}$$

Comparing this with $P(t)$ given in the problem, we calculate the lifetime and initial probability of the of the state. We have:

$$P(t) = P_0 e^{-\frac{2\Gamma}{\hbar}t} = e^{-\frac{t}{\tau}}$$
$$\therefore \begin{cases} P_0 = 1 \\ \frac{2\Gamma}{\hbar} = \frac{1}{\tau} \rightarrow \tau = \frac{\hbar}{2\Gamma} \end{cases}.$$