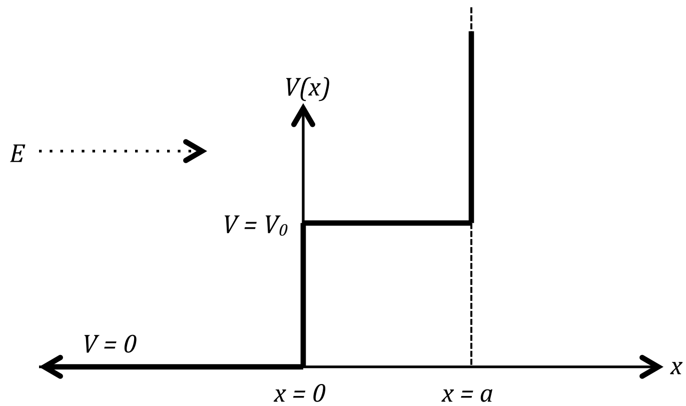
Physics 220
Homework #2
Spring 2017
Due Wednesday, 4/19/17

1. Consider reflection from a step potential of height V_0 with $E > V_0$ but now with an infinitely high wall added at a distance a from the step as shown below.
   a. What is $\psi(x)$ in each region?
   The wave functions in each region are given by:
   $$x<0:\ \psi(x) = Ae^{ikx} + Be^{-ikx};\quad k = \frac{\sqrt{2mE}}{\hbar}$$
   $$0<x<a:\ \psi(x) = Ce^{ik'x} + De^{-ik'x};\quad k' = \frac{\sqrt{2m(E-V_0)}}{\hbar}$$
   . Next we impose boundary conditions for the continuity of the wave function and its first derivative at $x=0$. We have
   $$\begin{aligned}&\psi @ x=0:\ \ A+B=C+D\\&\psi' @ x=0:\ \ ik(A-B)=ik'(C-D)\end{aligned}$$
   . Next we have one more boundary condition that the wave function must vanish at $x=a$. We have $\psi = 0 @ x=a:\ Ce^{ik'a} + De^{-ik'a} = 0$. We have three equations in four unknown coefficients so we can express the wave functions in terms of a single unknown amplitude A. Doing this we find:
   $$\begin{aligned}A &= \frac{C}{2}\left[1+\frac{k'}{k} - e^{2ik'a} + \frac{k'}{k}e^{2ik'a}\right]\\ B &= \frac{C}{2}\left[1-\frac{k'}{k} - e^{2ik'a} - \frac{k'}{k}e^{2ik'a}\right]\\ D &= -Ce^{2ik'a}\end{aligned}$$
   . Finishing the wave equation we have:
   $$x<0:\ \psi(x) = \frac{C}{2}\left[1+\frac{k'}{k} - e^{2ik'a} + \frac{k'}{k}e^{2ik'a}\right]e^{ikx} + \frac{C}{2}\left[1-\frac{k'}{k} - e^{2ik'a} - \frac{k'}{k}e^{2ik'a}\right]e^{-ikx}$$
   $$0<x<a:\ \psi(x) = Ce^{ik'x} - Ce^{2ik'a}e^{-ik'x}$$

   b. Show that the reflection coefficient at $x=0$ is $R=1$. This is different than the previously derived reflection coefficient without the infinite wall? What is the physical reason that $R=1$ in this case?
   $$R = \left(\frac{B}{A}\right)^{*}\frac{B}{A} = \left(\frac{\frac{C}{2}\left[1-\frac{k'}{k} - e^{-2ik'a} - \frac{k'}{k}e^{-2ik'a}\right]}{\frac{C}{2}\left[1+\frac{k'}{k} - e^{-2ik'a} + \frac{k'}{k}e^{-2ik'a}\right]}\right) \times \left(\frac{\frac{C}{2}\left[1-\frac{k'}{k} - e^{2ik'a} - \frac{k'}{k}e^{2ik'a}\right]}{\frac{C}{2}\left[1+\frac{k'}{k} - e^{2ik'a} + \frac{k'}{k}e^{2ik'a}\right]}\right) = 1$$
   After some easy algebra. The reason why the reflection coefficient has to be identically unity is that what ever may pass the barrier will be reflected back from the infinite wall; so all incident particles will be reflected.

c. Which part of the wave function represents a left moving particle at $x \le 0$? Show that this part of the wave function is an eigenfunction of the momentum operator and calculate the eigenvalue. Is the total wave function for $x \le 0$ an eigenfunction of the momentum operator?



The part of the wave function that represents the left moving particle for $x \le 0$ is given by Be^{-ikx}. To see if this is an eigenfunction of the momentum operator and determine the eigenvalue, we apply the momentum operator. We have $-i\hbar\frac{\partial}{\partial x}\left(Be^{-ikx}\right) = -i\hbar(-ik)Be^{-ikx} = -\hbar k Be^{-ikx}$ therefore the eigenvalue of the momentum operator is $-\hbar k$.

2. Griffith's 2.35

a. Let's write the wave functions in each region. We have:

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} \quad (x<0) \quad k = \dfrac{\sqrt{2mE}}{\hbar} \\ Fe^{ik'x} \quad (x<0) \quad k' = \dfrac{\sqrt{2m(E+V_0)}}{\hbar} \end{cases}$$

Continuity of ψ: $A + B = F$

Continuity of $\frac{d\psi}{dx}$: $ik(A-B) = ik'F$

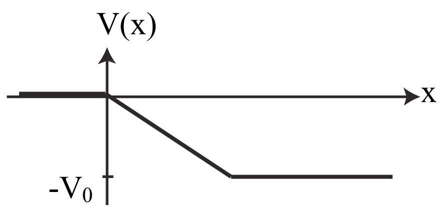
Next we solve eliminate the coefficient F between the two expressions and solve for the reflection coefficient:

$$A+B=\frac{k}{k'}(A-B)\rightarrow\frac{B}{A}=-\left(\frac{1-\frac{k}{k'}}{1+\frac{k}{k'}}\right)$$

$$R=\left(\frac{B}{A}\right)^{*}\frac{B}{A}=\left(\frac{1-\frac{k}{k'}}{1+\frac{k}{k'}}\right)^{2}=\left(\frac{k'-k}{k'+k}\right)^{2}=\left(\frac{\sqrt{E+V_0}-\sqrt{E}}{\sqrt{E+V_0}+\sqrt{E}}\right)^{2}=\left(\frac{\sqrt{1+\frac{V_0}{E}}-1}{\sqrt{1+\frac{V_0}{E}}+1}\right)^{2}$$

$$R=\left(\frac{\sqrt{1+3}-1}{\sqrt{1+3}+1}\right)^{2}=\left(\frac{1}{3}\right)^{2}=\frac{1}{9}$$

b. The cliff is two-dimensional, and even if we pretend the car drops straight down, the potential as a function of distance along the (crooked, but now one-dimensional) path is $-mgx$ (with x being the vertical coordinate), as shown.



c. In order to determine the reflection and transmission coefficients we need the ratio of $\frac{V_0}{E}=\frac{12MeV}{4MeV}=3$. Therefore the probability of being reflected is $R=\frac{1}{9}$ and transmitted is $R=\frac{8}{9}$.

3. Griffith's 2.52

    b. For the finite barrier we have from equations 2.167 and 2.168:

$$Eq.\ 2.167:\ \ B = i\frac{\sin(2k'a)}{2kk'}\left(k'^2 - k^2\right)F$$

$$Eq.\ 2.168:\ \ F = \frac{e^{-2ika}}{\cos(2k'a) - \frac{i\left(k'^2 + k^2\right)}{2kk'}\sin(2k'a)}A$$

$$\begin{pmatrix} B \\ F \end{pmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}\begin{pmatrix} A \\ G \end{pmatrix}$$

$$S = \frac{e^{-2ika}}{\cos(2k'a) - \frac{i\left(k'^2 + k^2\right)}{2kk'}\sin(2k'a)}\begin{bmatrix} i\frac{\sin(2k'a)}{2kk'}\left(k'^2 - k^2\right) & 1 \\ 1 & i\frac{\sin(2k'a)}{2kk'} \end{bmatrix}\begin{pmatrix} A \\ G \end{pmatrix}$$

4. Griffith's 2.53

    a. Writing the matrix given in the problem out, we have

$$F = M_{11}A + M_{12}B \qquad (1)$$

$$G = M_{21}A + M_{22}B \qquad (2)$$

and from the S matrix from the previous problem,

$$B = S_{11}A + S_{12}G \qquad (3)$$

$$F = S_{21}A + S_{22}G \qquad (4).$$

Solving (3) for G we and equating this to (2) we have

$G = \frac{B}{S_{12}} - \frac{S_{11}A}{S_{12}} = M_{21}A + M_{22}B$. Therefore we have $M_{22} = \frac{1}{S_{12}}$ and $M_{21} = -\frac{S_{11}}{S_{12}}$.

Next, we equate (1) and (4) and substitute the expression for G above and

$$F = S_{21}A + S_{22}G = S_{21}A + S_{22}\left(\frac{B}{S_{12}} - \frac{S_{11}A}{S_{12}}\right) = M_{11}A + M_{12}B$$

thus we have

$$M_{11}A + M_{12}B = \left(S_{21} - S_{22}\frac{S_{11}}{S_{12}}\right)A + \left(\frac{S_{22}}{S_{12}}\right)B.$$

Therefore we have $M_{12} = \frac{S_{22}}{S_{12}}$ and $M_{11} = \frac{S_{21}S_{12} - S_{22}S_{11}}{S_{12}}$.

Therefore the transfer matrix looks like:

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \frac{1}{S_{12}} \begin{bmatrix} S_{21}S_{12} - S_{22}S_{11} & S_{22} \\ -S_{11} & 1 \end{bmatrix}.$$

To write the scattering matrix in terms of the transfer matrix, we reverse the process. From equation (4) we solve for B and equate the expression to equation (1). We have $B = \frac{G}{M_{22}} - \frac{M_{21}}{M_{22}} A = S_{11}A + S_{12}G$. Therefore

$S_{11} = -\frac{M_{21}}{M_{22}}$ and $S_{12} = \frac{1}{M_{22}}$.

Next we equate equations (2) and (4) for F and use the expression for B in terms of G and A. We get

$$S_{21}A + S_{22}G = M_{11}A + M_{12}B = M_{11}A + M_{12}\left(\frac{G}{M_{22}} - \frac{M_{21}}{M_{22}} A \right).$$

$$S_{21}A + S_{22}G = \left(\frac{M_{11}M_{22} - M_{21}M_{12}}{M_{22}} \right) A + \frac{M_{12}}{M_{22}} G$$

Thus we have $S_{21} = \frac{M_{12}}{M_{22}}$ and $S_{22} = \frac{M_{11}M_{22} - M_{21}M_{12}}{M_{22}}$.

We can then write the scattering matrix as

$$S = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \frac{1}{S_{12}} \begin{bmatrix} S_{21}S_{12} - S_{22}S_{11} & S_{22} \\ -S_{11} & 1 \end{bmatrix}.$$

So to finish, we can write the reflection and transmission coefficients in terms of elements of the scattering and transfer matrices. The results are

$$R_{left} = \left| \frac{B}{A} \right|_{G=0} = |S_{11}|^2 = \left| \frac{M_{21}}{M_{22}} \right|^2$$

$$R_{right} = \left| \frac{F}{G} \right|_{A=0} = |S_{22}|^2 = \left| \frac{M_{11}M_{22} - M_{21}M_{12}}{M_{22}} \right|^2$$

$$T_{left} = \left| \frac{F}{A} \right|_{G=0} = |S_{21}|^2 = \left| \frac{M_{12}}{M_{22}} \right|^2$$

$$T_{right} = \left| \frac{B}{G} \right|_{A=0} = |S_{21}|^2 = \left| \frac{1}{M_{22}} \right|^2$$

b. From the figure given in the text, we have $\begin{pmatrix} C \\ D \end{pmatrix} = M_1 \begin{pmatrix} A \\ B \end{pmatrix}$ and $\begin{pmatrix} F \\ G \end{pmatrix} = M_2 \begin{pmatrix} C \\ D \end{pmatrix}$. Combining the two expressions we have $\begin{pmatrix} F \\ G \end{pmatrix} = M_2 \begin{pmatrix} C \\ D \end{pmatrix} = M_2 M_1 \begin{pmatrix} A \\ B \end{pmatrix} = M \begin{pmatrix} A \\ B \end{pmatrix}$. Thus we have $M = M_2 M_1$, which generalizes to any number of matrices and the order matters since we are multiplying matrices.

5. Starting with the expression for the transmission coefficient for the case of $E > V_0$, show that
   a. in the limit that $E \sim V_0$, the transmission coefficient can be written as $T \sim \dfrac{1}{1+a^2k^2}$.

   The transmission coefficient is given as $T^{-1} = 1 + \dfrac{V_0^2}{4E(E-V_0)} \sin^2\left(\sqrt{\dfrac{8ma^2}{\hbar^2}(E-V_0)}\right)$. For the first case, when $E > V_0$ we can write the transmission coefficient as approximately $T^{-1} \approx 1 + \dfrac{V_0}{4E(\frac{E}{V_0}-1)} \sin^2\left(\sqrt{\dfrac{8ma^2V_0}{\hbar^2}(\frac{E}{V_0}-1)}\right) \approx 1 + \dfrac{V_0^2}{4E^2}\sin^2\left(\sqrt{\dfrac{8ma^2E}{\hbar^2}}\right)$. For finite energies, expand the sine term in a power series and keep the lowest term. We have $\sin\theta = \theta + \dfrac{\theta^3}{3!} + \ldots \approx \theta$ and thus $\sin^2\left(\sqrt{\dfrac{8ma^2E}{\hbar^2}}\right) \sim \dfrac{8ma^2E}{\hbar^2}$. Therefore the transmission coefficient becomes

   $$T^{-1} \approx 1 + \frac{V_0^2}{4E^2}\sin^2\left(\sqrt{\frac{8ma^2E}{\hbar^2}}\right) \approx 1 + \frac{V_0^2}{4E^2}\left(\frac{8ma^2E}{\hbar^2}\right) \approx 1 + \left(k^2a^2\right)\frac{V_0^2}{E^2} \approx 1 + \left(k^2a^2\right)$$

   for $E \sim V_0$. And therefore we have $T \sim \dfrac{1}{1+\left(k^2a^2\right)}$.

   b. in the limit that $E < V_0$, the transmission coefficient can be written as $T = \dfrac{1}{1 + \dfrac{V_0^2}{4E(V_0-E)}\sinh^2\left(\sqrt{\dfrac{8ma^2}{\hbar^2}(V_0-E)}\right)}$. Hint, you will need to use the fact that $\sinh(x) = -i\sin(ix)$.

   Starting again with the transmission coefficient for $E > V_0$ we can write the transmission coefficient in the limit that $E < V_0$ as

$$T^{-1}=1+\frac{V_0^2}{4E^2(1-\frac{V_0}{E})}\sin^2\left(\sqrt{\frac{8ma^2E}{\hbar^2}(1-\frac{V_0}{E})}\right)=1+\frac{V_0^2}{4E^2(1-\frac{V_0}{E})}\sin^2\left(\sqrt{-\frac{8ma^2E}{\hbar^2}(\frac{V_0}{E}-1)}\right)$$

Now, we can write the sine term as

$$\sin\left(\sqrt{-\frac{8ma^2E}{\hbar^2}(\frac{V_0}{E}-1)}\right)=\sin\left(i\sqrt{\frac{8ma^2E}{\hbar^2}(\frac{V_0}{E}-1)}\right)=\frac{\sinh\left(\sqrt{\frac{8ma^2E}{\hbar^2}(\frac{V_0}{E}-1)}\right)}{-i}$$

using the definition that $\sinh(x)=-i\sin(x)$. Now square the sine (and hyperbolic sine terms) and we get $\sin^2(\theta)=\left(\frac{\sinh(\theta)}{-i}\right)\left(\frac{\sinh(\theta)}{i}\right)=\sinh^2(\theta)$. Therefore we can write the transmission coefficient for $E<V_0$ as

$$T^{-1}=1+\frac{V_0^2}{4E^2(1-\frac{V_0}{E})}\sinh^2\left(\sqrt{\frac{8ma^2}{\hbar^2}(V_0-E)}\right).$$

6. Suppose that you have a potential barrier of height $V_0=40eV$ and that a beam of electrons is incident on the barrier. At what incident energies greater than the barrier height will there be no reflected particles? That is, at what incident energies grater than the barrier height will $T=1$? Assume that the barrier has a width of $a=2.3\times10^{-10}m$ and determine the first 5 energies.

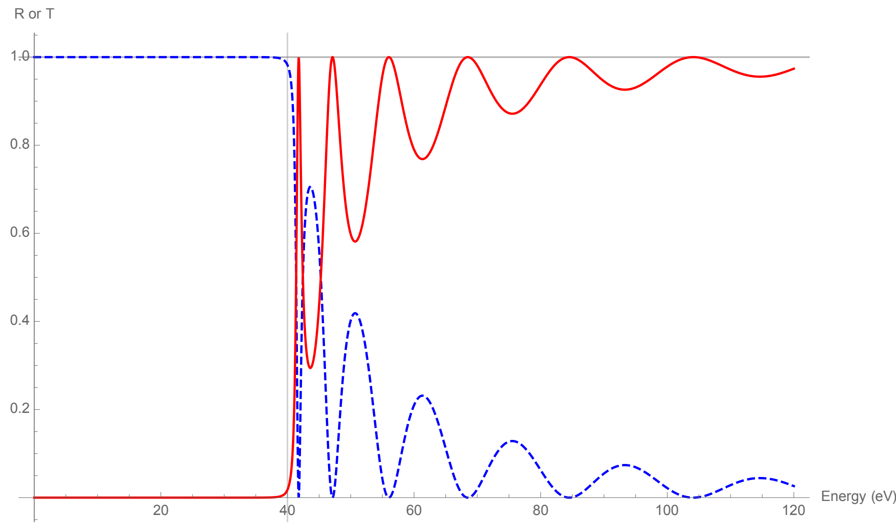
The transmission coefficient for a barrier of with $2a$, is given as

$$T=\frac{1}{1+\frac{V_0^2}{4E(E-V_0)}\sin^2\left(\sqrt{\frac{8ma^2}{\hbar^2}(E-V_0)}\right)}$$ and the transmission coefficient is unity when the argument of the sine squared term vanishes. Thus we have,

$$\sin^2\left(\sqrt{\frac{8ma^2}{\hbar^2}(E-V_0)}\right)=0\rightarrow\sqrt{\frac{8ma^2}{\hbar^2}(E-V_0)}=n\pi\,.$$

$$E_n=n^2\left(\frac{\pi^2\hbar^2}{8ma^2}\right)+V_0$$

For the data given in the problem, we find the first five energies are given as $E_n=\{41.8,47.1,56.0,68.5,84.5\}eV$. This can be seen also from the Mathematical plot below.



7. Fusion reactions are important in solar energy production and this process involves the capture of a proton by a carbon nucleus of radius about $2\times10^{-15}\,m$.
   a. What is the Coulomb potential experienced by the proton if it is at the nuclear surface of carbon? Express your answer in MeV.

$$V=\frac{Q_C Q_p}{4\pi\varepsilon_0 r}=\left[\frac{6\times\left(1.6\times10^{-19}C\right)^2}{4\pi\left(8.82\times10^{-12}\,\tfrac{C^2}{Nm^2}\right)\left(2\times10^{-15}\,m\right)}\right]\times\frac{1MeV}{1.6\times10^{-13}J}=4.32MeV$$

   b. The proton is incident upon the nucleus because of its thermal motion, which is given approximately as $E\sim10kT$, where the temperature in the interior of the sun is about $T\sim1\times10^{7}K$. How does this energy compare to the height of the Coulomb barrier?

$$E=10kT=10\times8.617\times10^{-5}\,\tfrac{eV}{K}\times1\times10^{7}K=8600eV=8.6keV$$

   c. Calculate the probability that the proton can penetrate a rectangular barrier potential of height V extending from r to $2r$, the point at which the Coulomb barrier potential drops to $\frac{V}{2}$. Hint: When we derived the transmission coefficient our barrier had a width of $2a$. In this problem the width is only a. You need to change the transmission coefficient appropriately to take this into account. You do not need to redo the analysis of the finite barrier if you're cleaver and think about it.

$$T = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)}\sinh^2\left(\sqrt{\frac{2ma^2}{\hbar^2}(V_0 - E)}\right)}$$

$$T = \frac{1}{1 + \frac{\left(4.32MeV\right)^2}{4\left(0.00866MeV\right)(4.32MeV)}\sinh^2\left(0.91\right)} = 0.0076 = 0.76\%$$

where, $\sinh^2\left(0.91\right)$ was evaluated using Mathematica. Further,

$$\sqrt{\frac{2ma^2}{\hbar^2}(V_0 - E)}$$

$$= \sqrt{\frac{2(1.67\times10^{-27}kg)\left(2\times10^{-15}m\right)^2 \times 4.32MeV}{\left(\frac{6.63\times10^{-34}Js}{2\pi}\right)^2} \times \frac{1.6\times10^{-13}J}{1MeV}} = 0.91 .$$