Physics 220
Homework #4
Spring 2017
Due Wednesday, 5/3/17

1. Griffith's 2.12
   a. The expectation value of the position is calculated from $\langle x \rangle = \langle \psi_n | x\psi_n \rangle$, where $x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$. In addition we need to operate on the wave function with the raising and lowering operators. From Griffith's page 48, equation 2.66, we have $a_+\psi_n = \sqrt{n+1}\psi_{n+1}$ and $a_-\psi_n = \sqrt{n}\psi_{n-1}$. Using these we compute the expectation value of the position.
   Thus $x\psi_n = \sqrt{\frac{\hbar}{2m\omega}}(a_+\psi_n + a_-\psi_n) = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\psi_{n+1} + \sqrt{n}\psi_{n-1}\right)$ and
   $\langle x \rangle = \langle \psi_n | x\psi_n \rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\int \psi_n^*\psi_{n+1}\,dx + \sqrt{n}\int \psi_n^*\psi_{n-1}\,dx\right] = 0$ since the states ψ_n and ψ_{n+1}, and ψ_n and ψ_{n-1} are orthogonal and thus the integral is zero.
   b. The expectation value of the momentum is calculated from $\langle p \rangle = \langle \psi_n | p\psi_n \rangle$, where $p = i\sqrt{\frac{m\omega\hbar}{2}}(a_+ - a_-)$. The expectation value of the momentum is thus
   $p\psi_n = i\sqrt{\frac{m\omega\hbar}{2}}(a_+\psi_n - a_-\psi_n) = i\sqrt{\frac{m\omega\hbar}{2}}\left(\sqrt{n+1}\psi_{n+1} - \sqrt{n}\psi_{n-1}\right)$ and
   $\langle p \rangle = \langle \psi_n | p\psi_n \rangle = i\sqrt{\frac{m\omega\hbar}{2}}\left[\sqrt{n+1}\int \psi_n^*\psi_{n+1}\,dx - \sqrt{n}\int \psi_n^*\psi_{n-1}\,dx\right] = 0$.
   c. The expectation value of the position squared is calculated from $\langle x^2 \rangle = \langle \psi_n | x^2\psi_n \rangle$, where $x^2 = \frac{\hbar}{2m\omega}(a_+ + a_-)(a_+ + a_-) = \frac{\hbar}{2m\omega}(a_+a_+ + a_+a_- + a_-a_+ + a_-a_-)$. The expectation value of the position squared is thus

$$x^2\psi_n = \frac{\hbar}{2m\omega}(a_+a_+\psi_n + a_+a_-\psi_n + a_-a_+\psi_n + a_-a_-\psi_n)$$

$$x^2\psi_n = \frac{\hbar}{2m\omega}\left(\sqrt{n+1}a_+\psi_{n+1} + \sqrt{n}a_+\psi_{n-1} + \sqrt{n+1}a_-\psi_{n+1} + \sqrt{n}a_-\psi_{n-1}\right)$$

$$x^2\psi_n = \frac{\hbar}{2m\omega}\left(\sqrt{n+1}\sqrt{n+2}\psi_{n+2} + \sqrt{n}\sqrt{n}\psi_n + \sqrt{n+1}\sqrt{n+1}\psi_n + \sqrt{n}\sqrt{n-1}\psi_{n-2}\right)$$

and

$$\left\langle x^2 \right\rangle = \left\langle \psi_n \middle| x^2 \psi_n \right\rangle$$

$$\left\langle x^2 \right\rangle = \frac{\hbar}{2m\omega}\left[\sqrt{n+1}\sqrt{n+2}\int \psi_n^* \psi_{n+2}\, dx + n\int \psi_n^* \psi_n\, dx + (n+1)\int \psi_n^* \psi_n\, dx + \sqrt{n}\sqrt{n-1}\int \psi_n^* \psi_{n-2}\, dx \right]$$

$$\left\langle x^2 \right\rangle = \frac{\hbar}{2m\omega}[0 + n + (n+1) + 0] = (2n+1)\frac{\hbar}{2m\omega}$$

d. The expectation value of the momentum squared is calculated from $\left\langle p^2 \right\rangle = \left\langle \psi_n \middle| p^2 \psi_n \right\rangle$, where $p^2 = -\frac{m\omega\hbar}{2}(a_+ - a_-)(a_+ - a_-) = -\frac{m\omega\hbar}{2}(a_+a_+ - a_+a_- - a_-a_+ + a_-a_-)$. The expectation value of the momentum squared is thus

$$p^2\psi_n = -\frac{m\omega\hbar}{2}\left(a_+a_+\psi_n - a_+a_-\psi_n - a_-a_+\psi_n + a_-a_-\psi_n\right)$$

$$p^2\psi_n = -\frac{m\omega\hbar}{2}\left(\sqrt{n+1}a_+\psi_{n+1} - \sqrt{n}a_+\psi_{n-1} - \sqrt{n+1}a_-\psi_{n+1} + \sqrt{n}a_-\psi_{n-1}\right)$$

$$p^2\psi_n = -\frac{m\omega\hbar}{2}\left(\sqrt{n+1}\sqrt{n+2}\psi_{n+2} - \sqrt{n}\sqrt{n}\psi_n - \sqrt{n+1}\sqrt{n+1}\psi_n + \sqrt{n}\sqrt{n-1}\psi_{n-2}\right)$$

and

$$\left\langle p^2 \right\rangle = \left\langle \psi_n \middle| p^2 \psi_n \right\rangle$$

$$\left\langle p^2 \right\rangle = -\frac{m\omega\hbar}{2}\left[\sqrt{n+1}\sqrt{n+2}\int \psi_n^* \psi_{n+2}\, dx - n\int \psi_n^* \psi_n\, dx - (n+1)\int \psi_n^* \psi_n\, dx + \sqrt{n}\sqrt{n-1}\int \psi_n^* \psi_{n-2}\, dx \right]$$

$$\left\langle p^2 \right\rangle = \frac{m\omega\hbar}{2}[0 + n + (n+1) + 0] = (2n+1)\frac{m\omega\hbar}{2}$$

e. The problem also asks to calculate the expectation value of the kinetic energy. I'm also going to calculate a few other things. In particular the expectation value of the potential energy and the expectation value of the Hamiltonian.

$$\left\langle T \right\rangle = \frac{\left\langle p^2 \right\rangle}{2m} = (2n+1)\frac{m\omega\hbar}{4m} = (2n+1)\frac{\hbar\omega}{4}.$$

$$\left\langle V \right\rangle = \frac{m\omega^2}{2}\left\langle x^2 \right\rangle = (2n+1)\frac{m\omega^2\hbar}{4m\omega} = (2n+1)\frac{\hbar\omega}{4}.$$

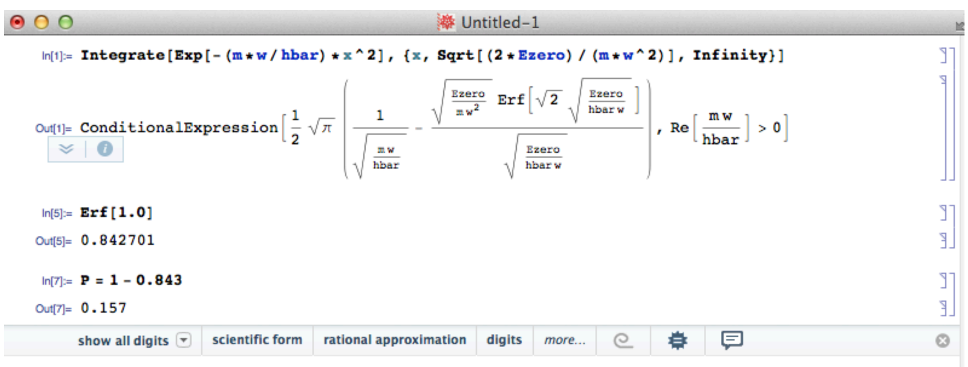
$$\left\langle H \right\rangle = \left\langle T \right\rangle + \left\langle V \right\rangle = (2n+1)\left[\frac{\hbar\omega}{4} + \frac{\hbar\omega}{4}\right] = (2n+1)\frac{\hbar\omega}{2} = \left(n + \frac{1}{2}\right)\hbar\omega \text{ as expected!}$$

2. Griffith's 2.15

The ground state wave function is $\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$ and the classically allowed region is given by $E_0 = \frac{1}{2}m\omega^2 x_0^2 \rightarrow x_0 = \sqrt{\frac{2E_0}{m\omega^2}}$. To calculate the probability we use

$$P = 2\left[\int_{x_0}^{\infty} \psi_0^* \psi_0 \, dx\right]\psi_0 = 2\left[\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \int_{x_0}^{\infty} e^{-\frac{m\omega}{\hbar}x^2} \, dx\right],$$

where the factor of two is from integrating from x_0 to infinity and from minus infinity to x_0 outside of the classically allowed region. Evaluating the integral on Mathematica (or looking it up in a table of integrals) we find: $P = 2\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}\left[\frac{1}{2}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} - \frac{1}{2}\sqrt{\frac{\pi\hbar\omega E_0}{m\omega^2 E_0}} Erf[\sqrt{\frac{2E_0}{\hbar\omega}}]\right] = [1 - Erf[1]]$ using the fact that the ground state energy is $E_0 = \frac{\hbar\omega}{2}$. Evaluating the error function on Mathematica we find that the probability is given as $P = 1 - Erf[1] = 1 - 0.843 = 0.157$, or a 15.7% chance of being found outside of the classically forbidden region! The Mathematica code is given below.



3. Prove that $\hat{H}\left(\hat{a}_-|\psi_n\rangle\right) = \left(E_n - \hbar\omega\right)|\psi_{n-1}\rangle$.

Starting with $\hat{H}\hat{a}_-|\psi_n\rangle = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)\hat{a}_-|\psi_n\rangle$. Multiply through by the lowering operator on the right and we have $\hbar\omega\left(\hat{a}_-\hat{a}_+\hat{a}_- - \frac{1}{2}\hat{a}_-\right)|\psi_n\rangle$. Factor out the lowering operator on the left and replace $\hat{a}_+\hat{a}_-$ with $\hat{a}_-\hat{a}_+ - 1$. We have $\hbar\omega\hat{a}_-\left(\hat{a}_+\hat{a}_- - \frac{1}{2}\right)|\psi_n\rangle = \hbar\omega\hat{a}_-\left(\hat{a}_-\hat{a}_+ - 1 - \frac{1}{2}\right)|\psi_n\rangle$. Then we note that $\hat{H} = \left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)\hbar\omega$, so we can write $\hbar\omega\hat{a}_-\left(\frac{\hat{H}}{\hbar\omega} - 1\right)|\psi_n\rangle = \hat{a}_-\left(\hat{H}|\psi_n\rangle - \hbar\omega|\psi_n\rangle\right) = \left(E_n - \hbar\omega\right)\hat{a}_-|\psi_n\rangle$.

Therefore, $\hat{H}\left(\hat{a}_-|\psi_n\rangle\right) = \left(E_n - \hbar\omega\right)\hat{a}_-|\psi_n\rangle = \left(E_n - \hbar\omega\right)|\psi_{n-1}\rangle$.

4. Starting from $|\psi_0\rangle$, use the raising operator to determine $|\psi_2\rangle$. Don't forget to normalize your solution. Then, using the analytic solution to the harmonic oscillator $|\psi_n\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} e^{-\frac{m\omega}{2\hbar}x^2} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$ show that your solution using the raising operator for $|\psi_2\rangle$ agrees with the analytic solution.

Using the general analytic solution to the harmonic oscillator we can form $|\psi_2\rangle$. Thus we have:

$$|\psi_2\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^2 2!}} e^{-\frac{m\omega}{2\hbar}x^2} H_2\left(\sqrt{\frac{m\omega}{\hbar}}x\right) = \frac{1}{\sqrt{8}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2} H_2\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$$

and we need to evaluate the Hermite polynomial of order 2.

$$H_2\left(\sqrt{\frac{m\omega}{\hbar}}x\right) = \sum_n a_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)^n = a_0 + a_2\left(\sqrt{\frac{m\omega}{\hbar}}x\right)^2$$

where the coefficient a_2 is determined from the recursion relation.

$$a_{n+2} = \left[\frac{2n-\lambda}{(n+2)(n+1)}\right]a_n \rightarrow a_2 = -\frac{\lambda}{2}a_0$$

Thus $H_2\left(\sqrt{\frac{m\omega}{\hbar}}x\right) = a_0 - \frac{\lambda}{2}a_0\left(\sqrt{\frac{m\omega}{\hbar}}x\right)^2$. The unknown coefficient a_0 is determined by setting the coefficient in front of the highest power of x^2 equal to $2^2 = 4$. We have $-\frac{\lambda}{2}a_0 = 4 \rightarrow a_0 = -\frac{8}{\lambda} = -\frac{8}{4} = -2$, where $\lambda = \frac{2E_2}{\hbar\omega} - 1 = \frac{2\left(\frac{5}{2}\hbar\omega\right)}{\hbar\omega} - 1 = 4$.
$\lambda = \frac{2E_2}{\hbar\omega} - 1 = \frac{2\left(\frac{5}{2}\hbar\omega\right)}{\hbar\omega} - 1 = 4$. Evaluating the Hermite polynomial we have $H_2\left(\sqrt{\frac{m\omega}{\hbar}}x\right) = -2 + 4\frac{m\omega}{\hbar}x^2 = \left(4\frac{m\omega}{\hbar}x^2 - 2\right)$. This could also be evaluated on Mathematica. The code is below.

```
In[1]:= HermiteH[2, Sqrt[m*w/hbar]*x]

Out[1]= -2 + 4 m w x^2 / hbar
```

So the analytic solution is

$$|\psi_2\rangle = \frac{1}{\sqrt{8}}\left(\frac{4m\omega}{\hbar}x^2 - 2\right)\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2} = \frac{1}{\sqrt{2}}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)|\psi_0\rangle$$

Using the raising operator, $a_+ = \frac{1}{\sqrt{2m\hbar\omega}}(-ip + m\omega x)$ we will raise $|\psi_0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$ to $|\psi_1\rangle$ and then $|\psi_1\rangle$ to $|\psi_2\rangle$. In the momentum operator we need to evaluate $\frac{d}{dx}|\psi_0\rangle$, which is $\frac{d}{dx}|\psi_0\rangle = -\frac{m\omega}{\hbar}x|\psi_0\rangle$. Applying the raising operator $|\psi_1\rangle = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x + m\omega x)|\psi_0\rangle = \sqrt{\frac{2m\omega}{\hbar}}x|\psi_0\rangle$. Now we raise $|\psi_1\rangle$ to $|\psi_2\rangle = a_+ A_2|\psi_1\rangle$. Evaluating the derivative in the momentum operator

$$\frac{d}{dx}|\psi_1\rangle = \sqrt{\frac{2m\omega}{\hbar}}\frac{d}{dx}(x|\psi_0\rangle) = \sqrt{\frac{2m\omega}{\hbar}}\left(|\psi_0\rangle + x\frac{d}{dx}|\psi_0\rangle\right) = \sqrt{\frac{2m\omega}{\hbar}}\left(1 - \frac{m\omega}{\hbar}x^2\right)|\psi_0\rangle.$$

Now applying the raising operator

$$|\psi_2\rangle = \frac{A_2}{\sqrt{2m\hbar\omega}}\left[-\hbar\sqrt{\frac{2m\omega}{\hbar}}\left(1 - \frac{m\omega}{\hbar}x^2\right)|\psi_0\rangle + \sqrt{\frac{2m\omega}{\hbar}}m\omega x^2|\psi_0\rangle\right]$$

$$|\psi_2\rangle = \frac{A_2\hbar}{\sqrt{2m\hbar\omega}}\sqrt{\frac{2m\omega}{\hbar}}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)|\psi_0\rangle$$

$$|\psi_2\rangle = A_2\left(\frac{2m\omega}{\hbar}x^2 - 1\right)|\psi_0\rangle$$

Now we need to normalize the solution to determine A_2. Normalizing we find that

$$P = 1 = \langle\psi_2|\psi_2\rangle = A_2^2\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)^2 e^{-\frac{m\omega}{\hbar}x^2}dx = 2A_2^2 \to A_2 = \frac{1}{\sqrt{2}}.$$

This integral was done on Mathematica. The code is below. Thus the normalized wave function is $|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)|\psi_0\rangle$, which agrees with the analytic solution.

```
In[2]:= Integrate[A2^2*Sqrt[(m*w)/(Pi*hbar)]*((2*m*w/hbar)*x^2-1)^2*
          Exp[-m*w*x^2/hbar], {x, -Infinity, Infinity}]

Out[2]= ConditionalExpression[2 A2^2, Re[m w / hbar] > 0]
```

5. Consider a charged particle of mass m and charge q in a one-dimensional harmonic oscillator potential. Suppose that an electric field E is turned on so that the potential energy is given by $V = \frac{m\omega^2}{2}x^2 - q\mathrm{E}x$. What are the energies of the states? Hint: The problem is easier with a change of variables and thus let $y = x - \frac{q\mathrm{E}}{m\omega^2}$.

We start with the SWE and use the hint. The SWE is for the harmonic oscillator in the presence of zero electric field is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2}+V\psi_n=E_n\psi_n$$
$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2}+\frac{m\omega^2}{2}x^2\psi_n=E_n\psi_n$$
where the energies are given as $E_n=\left(n+\frac{1}{2}\right)\hbar\omega$.

In the presence of the electric field, the SWE is
$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2}+V\psi_n=E_n^{'}\psi_n$$
$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2}+\frac{m\omega^2}{2}x^2\psi_n-q\mathrm{E}x\psi_n=E_n^{'}\psi_n$$
Using the hint, we can write that $y=x-\frac{q\mathrm{E}}{m\omega^2}\rightarrow dy=dx$, and $x=y+\frac{q\mathrm{E}}{m\omega^2}$ so that the SWE becomes

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2}+V\psi_n=E_n^{'}\psi_n$$
$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2}+\left(\frac{m\omega^2}{2}x^2-q\mathrm{E}x\right)\psi_n=-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2}+\frac{m\omega^2}{2}x\left(x-\frac{q\mathrm{E}}{m\omega^2}-\frac{q\mathrm{E}}{m\omega^2}\right)\psi_n=E_n^{'}\psi_n$$
$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dy^2}+\frac{m\omega^2}{2}\left(y+\frac{q\mathrm{E}}{m\omega^2}\right)\left(y-\frac{q\mathrm{E}}{m\omega^2}\right)\psi_n=E_n^{'}\psi_n$$
$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dy^2}+\frac{m\omega^2}{2}y^2\psi_n-\frac{q^2\mathrm{E}^2}{2m\omega^2}\psi_n=E_n^{'}\psi_n$$
$$\therefore-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dy^2}+\frac{m\omega^2}{2}y^2\psi_n=\left(E_n^{'}+\frac{q^2\mathrm{E}^2}{2m\omega^2}\right)\psi_n$$
This form of the SWE for the harmonic oscillator looks exactly like the original form and therefore we have
$$E_n=E_n^{'}+\frac{q^2\mathrm{E}^2}{2m\omega^2}$$
$$\left(n+\frac{1}{2}\right)\hbar\omega=E_n^{'}+\frac{q^2\mathrm{E}^2}{2m\omega^2}$$
$$\therefore E_n^{'}=\left(n+\frac{1}{2}\right)\hbar\omega-\frac{q^2\mathrm{E}^2}{2m\omega^2}$$
and the energies of the states are those of the original harmonic oscillator lowered by a constant $\frac{q^2\mathrm{E}^2}{2m\omega^2}$.

6. Griffith's 3.10
   The ground state wave function of the infinite square well is given as
   $\psi_1 = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right)$. Apply the momentum operator and we have
   $\hat{p}\psi_1 = -i\hbar \frac{d}{dx}\left(\sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right)\right) = -i\hbar\sqrt{\frac{2}{a}}\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}x\right)$. Therefore since we do not get the wave function back multiplied by a constant, the ground state wave function of the infinite square well is not an eigenstate of the momentum operator.

7. Griffith's 3.13

   a.
   $$[AB,C] = ABC - CAB = ABC - CAB + (ACB - ACB)$$
   $$[AB,C] = A[B,C] - [C,A]B = A[B,C] + [A,C]B$$

   b.
   $$[x^n, p] \Rightarrow [x^n, p]f = -i\hbar x^n \frac{df}{dx} - (-i\hbar \frac{d}{dx}(x^n f)) = -i\hbar x^n \frac{df}{dx} + i\hbar\left(nx^{n-1}f + x^n \frac{df}{dx}\right).$$
   $$[x^n, p]f = i\hbar n x^{n-1} f \rightarrow [x^n, p] = i\hbar n x^{n-1}$$

   c.
   $$[f, p] \Rightarrow [f, p]q = fpq - pfq = -i\hbar f \frac{dq}{dx} - (-i\hbar \frac{d}{dx}(fq))$$
   $$= -i\hbar f \frac{dq}{dx} + i\hbar(\frac{df}{dx}q + f\frac{dq}{dx})$$
   $$[f, p]q = i\hbar \frac{df}{dx} q \rightarrow [f, p] = i\hbar \frac{df}{dx}$$