Physics 220
Homework #6
Spring 2017
Due Wednesday, 5/17/17

1. An electron in the ground state of tritium, for which the nucleus consists of a proton and two neutrons. A nuclear reaction instantaneously changes the nucleus to He^3, that is, two protons and one neutron. Calculate the probability that the electron remains in the ground state of He^3.

   From class, we had the expression $\frac{Zme^2}{4\pi\varepsilon_0\hbar^2}$. We set $Z=1$ for hydrogen and define the Bohr radius as $a=\frac{me^2}{4\pi\varepsilon_0\hbar^2}$. Leaving Z for any one-electron atom, we can replace the Bohr radius in the hydrogen wave functions with $\frac{a}{Z}$. The ground state of hydrogen and helium are $|\psi_{100}\rangle_H = \frac{1}{\sqrt{\pi a^3}} e^{-\frac{r}{a}}$ and $|\psi_{100}\rangle_{He} = \frac{1}{\sqrt{\pi\left(\frac{a}{Z}\right)^3}} e^{-\frac{r}{a/Z}} = \sqrt{\frac{8}{\pi a^3}} e^{-\frac{2r}{a}}$ respectively.

   The probability that helium is found in the ground state is given by

$$\langle \psi_{100,H} | \psi_{100,He} \rangle = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^{\infty} \frac{1}{\sqrt{\pi a^3}} e^{-\frac{r}{a}} \sqrt{\frac{8}{\pi a^3}} e^{-\frac{2r}{a}} r^2 dr = \frac{4\pi\sqrt{8}}{\pi a^3} \int_0^{\infty} e^{-\frac{3r}{a}} r^2 \, dr .$$

$$\langle \psi_{100,H} | \psi_{100,He} \rangle = \frac{4\pi\sqrt{8}}{\pi a^3} \left[\frac{2a^3}{27} \right] = 0.838 = 83.3\%$$

2. Griffith's 4.19
   a. The commutator of the z-component of the angular momentum with the coordinates are:

$$[L_z, x] = [xp_y - yp_x, x] = xp_y x - yp_x x - xxp_y + xyp_x = x[p_y, x] + y[x, p_x] = i\hbar y$$

$$[L_z, y] = [xp_y - yp_x, y] = xp_y y - yp_x y - yxp_y + yyp_x = x[p_y, y] + y[y, p_x] = -i\hbar x$$

$$[L_z, z] = [xp_y - yp_x, z] = xp_y z - yp_x z - zxp_y + zyp_x = x[p_y, z] + y[z, p_x] = 0$$

   where we have used the results from Griffith's problem 4.1, where

$$[r_i, p_j] = -[p_i, r_j] = i\hbar\delta_{ij} = i\hbar \begin{cases} 0 & if \;\; i \neq j \\ 1 & if \;\; i = j \end{cases} .$$

   The commutator of the z-component of the angular momentum with the momentum operators are:

$$[L_z, p_x] = [xp_y - yp_x, p_x] = [xp_y, p_x] - [yp_x, p_x]$$
$$= \{x[p_y, p_x] + [x, p_x]p_y\} - \{y[p_x, p_x] + [y, p_x]p_x\}$$
$$= 0 + i\hbar p_y - 0 - 0$$
$$[L_z, p_x] = i\hbar p_y$$
$$[L_z, p_y] = [xp_y - yp_x, py_x] = [xp_y, p_y] - [yp_x, p_y]$$
$$= \{x[p_y, p_y] + [x, p_y]p_y\} - \{y[p_x, p_y] + [y, p_y]p_x\}$$
$$= 0 + 0 - 0 - i\hbar p_x$$
$$[L_z, p_y] = -i\hbar p_x$$
$$[L_z, p_z] = [xp_y - yp_x, p_z] = [xp_y, p_z] - [yp_x, p_z]$$
$$= \{x[p_y, p_z] + [x, p_z]p_y\} - \{y[p_x, p_z] + [y, p_z]p_x\}$$
$$= 0 + 0 - 0 - 0$$
$$[L_z, p_x] = 0$$

where in both parts, we've used from homework #4, problem Griffith's 3.13, where $[AB, C] = A[B, C] + [A, C]B$

b. In order to do this part, we require two relationships between commutators. The first is a footnote from page 160, $[A, B + C] = [A, B] + [A, C]$ and the second is from homework #4, problem Griffith's 3.13, where $[AB, C] = A[B, C] + [A, C]B$ and by analogy $[A, BC] = [A, B]C + B[A, C]$. Thus,

$$[L_z, L_x] = [L_z, yp_z - zp_y] = [L_z, yp_z] - [L_z, zp_y]$$
$$[L_z, L_x] = \{[L_z, y]p_z + y[L_z, p_z]\} - \{[L_z, z]p_y + z[L_z, p_y]\}$$
$$[L_z, L_x] = \{-i\hbar xp_z + 0\} - \{0 - zi\hbar p_x\}$$
$$[L_z, L_x] = -i\hbar\left(xp_z - zp_x\right)$$
$$[L_z, L_x] = i\hbar L_y$$

where we used $[A, BC] = [A, B]C + B[A, C]$ again.

c. The commutator of the z-component of the angular momentum and the r^2 operator is:

$$[L_z, r^2] = \{L_z, x^2 + y^2 + z^2] = [L_z, x^2] + [L_z, y^2] + [L_z, z^2]$$
$$[L_z, r^2] = [L_z, xx] + [L_z, yy] + [L_z, zz]$$
$$[L_z, r^2] = \left\{[L_z, x]x + x[L_z, x]\right\} + \left\{[L_z, y]y + y[L_z, y]\right\} + \left\{[L_z, z]z + z[L_z, z]\right\}$$
$$[L_z, r^2] = \left\{i\hbar yx + i\hbar yx\right\} + \{-i\hbar xy - i\hbar yx\} + \{0 + 0\}$$
$$[L_z, r^2] = 0$$

The commutator of the z-component of the angular momentum and the p^2 operator is:

$$[L_z, p^2] = \{L_z, p_x^2 + p_y^2 + p_z^2] = [L_z, p_x^2] + [L_z, p_y^2] + [L_z, p_z^2]$$
$$[L_z, p^2] = [L_z, p_x p_x] + [L_z, p_y p_y] + [L_z, p_z p_z]$$
$$[L_z, p^2] = \left\{[L_z, p_x]p_x + p_x[L_z, p_x]\right\} + \left\{[L_z, p_y]p_y + p_y[L_z, p_y]\right\} + \left\{[L_z, p_z]p_z + p_z[L_z, p_z]\right\}$$
$$[L_z, p^2] = \left\{i\hbar p_y p_x + i\hbar p_x p_y\right\} + \{-i\hbar p_x p_y - i\hbar p_y p_x\} + \{0 + 0\}$$
$$[L_z, p^2] = 0$$

d. Since p^2 and r^2 commute with each component of L, L commutes with the Hamiltonian.

3. Griffith's 4.22

   a. Since the largest value of m_l is l, there is no state $m_l = l+1$. Thus $L_+ Y_l^l = 0$.

   b. The raising operator is given in equation 4.130 by $L_+ = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$. Acting with this we have $L_+ Y_l^l = 0 = \hbar e^{i\phi}\left(\frac{\partial Y_l^l}{\partial\theta} + i\cot\theta\frac{\partial Y_l^l}{\partial\phi}\right)$. Using the fact that $L_z Y_l^l = -i\hbar\frac{\partial Y_l^l}{\partial\phi} = l\hbar Y_l^l$ we have that $\int\frac{dY_l^l}{Y_l^l} = il\int d\phi \to \ln Y_l^l = il\phi + C' \to Y_l^l = Ce^{il\phi}$. The constant C may be a constant independent of ϕ, but it could have a θ dependence. Using this result in the expression with the raising operator we have:

   $$0 = \left(\frac{\partial Ce^{il\phi}}{\partial\theta} + i\cot\theta\frac{\partial Ce^{il\phi}}{\partial\phi}\right) = e^{il\phi}\frac{dC}{d\theta} - l\cot\theta Ce^{il\phi} \to \int\frac{dC}{C} = l\int\cot\theta\, d\theta \to \ln C = l\ln\sin\theta + K$$

   Next we solve this for C and find: $C = A\sin^l\theta$. Thus up to the normalization we have $Y_l^l = A\sin^l\theta e^{il\phi}$

   c. Normalizing this solution we have:

   $$P = 1 = A^2\int_0^{\pi}\sin^{2l}\theta\sin\theta\, d\theta\int_0^{2\pi}d\phi = A^2\left[\frac{\sqrt{\pi}\,\Gamma(l+1)}{\Gamma\left(l+\frac{3}{2}\right)}\right][2\pi] \to A = \left(\frac{1}{\sqrt{4\pi^3}}\right)^{\frac{1}{2}}\sqrt{\frac{\Gamma\left(l+\frac{3}{2}\right)}{\Gamma(l+1)}},$$ where

   $\Gamma\left(l+\frac{3}{2}\right)$ and $\Gamma(l+1)$ are gamma functions.

   The Mathematica code is below.

```
Integrate[Sin[t]^(2 l + 1), {t, 0, Pi}]

Out[23]= ConditionalExpression[ (√π Gamma[1 + l]) / (Gamma[3/2 + l]) , Re[l] > -1]
```

4. Griffith's 4.44 part c

   $$\left(L_x^2 + L_y^2\right)|\psi_{433}\rangle = \left(L^2 - L_z^2\right)|\psi_{433}\rangle = L^2|\psi_{433}\rangle - L_z^2|\psi_{433}\rangle$$
   $$\left(L_x^2 + L_y^2\right)|\psi_{433}\rangle = 3(3+1)\hbar^2|\psi_{433}\rangle - (3\hbar)^2|\psi_{433}\rangle = 3\hbar^2|\psi_{433}\rangle$$
   $$\left(L_x^2 + L_y^2\right)|\psi_{433}\rangle == 3\hbar^2|\psi_{433}\rangle$$

   and thus $\left(L_x^2 + L_y^2\right) = 3\hbar^2$. Since this is the only state, the probability of returning this value is unity.

5. Consider the hydrogen atom wave function $|\psi_{432}\rangle$. What are

   a. What is the total energy in electron volts?

   The energy is given by $E_n = -\frac{13.6eV}{n^2} = -\frac{13.6eV}{4^2} = -0.85eV$.

   b. What is the expectation value of the radial coordinate?

   The expectation value of the radial coordinate is $\langle r\rangle = \langle\psi_{432}|r\psi_{432}\rangle$. Evaluating:

$$\langle r\rangle = \int_0^\infty \left(\frac{1}{768\sqrt{35}} a^{-\frac{3}{2}} \left(\frac{r}{a}\right)^3 e^{-\frac{r}{4a}}\right)^2 r r^2\, dr \int_0^\pi \left(\sqrt{\frac{105}{32\pi}} \sin^2\theta \cos\theta\right)^2 \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$\langle r\rangle = \frac{5.06\times 10^{-8}}{a^9}\left[3.72\times 10^8 a^{10}\right]\left[\frac{16}{105}\right]\left[2\pi\right] = 18a$$

The Mathematica code is shown below for the integrals.

```
In[22]:= Integrate[r^9*Exp[-r/(2*a)], {r, 0, Infinity}]
Out[22]= ConditionalExpression[371 589 120 a^10, Re[a] > 0]

In[21]:= Integrate[Sin[t]^5*Cos[t]^2, {t, 0, Pi}]
Out[21]= 16/105
```

c. What is the total angular momentum?
The total angular momentum is
$L|\psi_{nlm}\rangle = \sqrt{l(l+1)}\hbar|\psi_{nlm}\rangle \rightarrow L|\psi_{432}\rangle = \sqrt{3(3+1)}\hbar|\psi_{432}\rangle = \sqrt{12}\hbar|\psi_{432}\rangle$ which is $\sqrt{12}\hbar$.

d. What is the z-component of the total angular momentum?
The z-component of the angular momentum is
$L_z|\psi_{nlm}\rangle = m_l\hbar|\psi_{nlm}\rangle \rightarrow L_z|\psi_{432}\rangle = 2\hbar|\psi_{432}\rangle$, which is $2\hbar$.