Physics 220
Homework #7
Spring 2017
Due Wednesday, 5/24/17

1. Show that the lowering matrix for spin is given by $S_- = \hbar\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

   We define the S_- matrix as $S_- = \begin{bmatrix} S_{-11} & S_{-12} \\ S_{-21} & S_{-22} \end{bmatrix}$, where the matrix elements are given as:

   $S_{-11} = \langle\uparrow|S_-\uparrow\rangle$

   $S_{-12} = \langle\uparrow|S_-\downarrow\rangle$

   $S_{-21} = \langle\downarrow|S_-\uparrow\rangle$

   $S_{-22} = \langle\downarrow|S_-\downarrow\rangle$

   To evaluate the matrix elements we need to evaluate $S_-|\uparrow\rangle$ and $S_-|\downarrow\rangle$ respectively. We have $S_-|\uparrow\rangle = \hbar\sqrt{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)-\tfrac{1}{2}\left(\tfrac{1}{2}-1\right)}|\downarrow\rangle = \hbar|\downarrow\rangle$ and $S_-|\downarrow\rangle = 0$ (since the spin down state cannot be lowered any more.

   Thus,

   $S_{-11} = \langle\uparrow|S_-\uparrow\rangle = \hbar\langle\uparrow|\downarrow\rangle = 0$

   $S_{-12} = \langle\uparrow|S_-\downarrow\rangle = 0\langle\uparrow|\downarrow\rangle = 0$

   $S_{-21} = \langle\downarrow|S_-\uparrow\rangle = \hbar\langle\downarrow|\downarrow\rangle = \hbar$

   $S_{-22} = \langle\downarrow|S_-\downarrow\rangle = 0\langle\downarrow|\downarrow\rangle = 0$

   Therefore, $S_- = \begin{bmatrix} S_{-11} & S_{-12} \\ S_{-21} & S_{-22} \end{bmatrix} = \hbar\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

2. Griffith's 4.25
   The spin of the electron is given from

   $$S = I\omega = \tfrac{2}{5}mr^2\left(\tfrac{v}{r}\right) \rightarrow v = \frac{5S}{2mr} = \frac{5\hbar 4\pi\varepsilon_0 mc^2}{4me^2} = \frac{5\hbar\pi\varepsilon_0 c^2}{e^2}$$

   $$v = \frac{5\hbar\pi\varepsilon_0 c^2}{e^2} = \frac{5\pi\left(\tfrac{6.63}{2\pi}\times10^{-34}\,Js\right)\left(8.85\times10^{-12}\,\tfrac{C^2}{Nm^2}\right)\left(3\times10^8\,\tfrac{m}{s}\right)^2}{\left(1.6\times10^{-19}C\right)^2}$$

   $$v = 5.2\times10^{10}\,\tfrac{m}{s}$$

   and this is greater than the speed of light, which is not possible.

3. Griffith's 4.29

a. The eigenspinors are determined (following the examples in class) as:

$$S_y\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=c\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

$$\rightarrow\begin{cases}-i\dfrac{\hbar}{2}\psi_2=c\psi_1\rightarrow\psi_1=-i\dfrac{\hbar}{2c}\psi_2\\ i\dfrac{\hbar}{2}\psi_1=c\psi_2\rightarrow i\dfrac{\hbar}{2}\left(-i\dfrac{\hbar}{2c}\psi_2\right)=c\psi_2\rightarrow c=\pm\dfrac{\hbar}{2}\end{cases}$$

$$\therefore\psi_2=\mp i\psi_1$$

So, we define two states (the eigenspinors) $|\nearrow\rangle=c'\begin{pmatrix}1\\ -i\end{pmatrix}$ and $|\swarrow\rangle=c'\begin{pmatrix}1\\ i\end{pmatrix}$ with eigenvalues $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ respectively. We need to normalize these states. The normalization for each state is given as:

$$1=\langle\swarrow|\swarrow\rangle=c'^*\begin{pmatrix}1 & -i\end{pmatrix}c'\begin{pmatrix}1\\ i\end{pmatrix}=2|c'|^2\rightarrow c'=\frac{1}{\sqrt{2}}\text{ and}$$

$$1=\langle\nearrow|\nearrow\rangle=c'^*\begin{pmatrix}1 & i\end{pmatrix}c'\begin{pmatrix}1\\ -i\end{pmatrix}=2|c'|^2\rightarrow c'=\frac{1}{\sqrt{2}}.$$

Thus the normalized eigenspinors are:

$$|\nearrow\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\text{ and }|\swarrow\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}.$$

b. The general state is given by $\psi=\begin{pmatrix}a\\ b\end{pmatrix}$. The probability of a measurement on each of the states is given by:

$$P_+=|\langle\nearrow|\psi\rangle|^2=\left|\frac{1}{\sqrt{2}}(1-i)^*\begin{pmatrix}a\\ b\end{pmatrix}\right|^2=\frac{1}{2}(a+ib)^2$$

$$P_-=|\langle\swarrow|\psi\rangle|^2=\left|\frac{1}{\sqrt{2}}(1+i)^*\begin{pmatrix}a\\ b\end{pmatrix}\right|^2=\frac{1}{2}(a-ib)^2$$

The sum of the probabilities is:

$$P=P_++P_-=\frac{1}{2}\left[(a+ib)^2+(a-ib)^2\right]=\frac{1}{2}\left[(a+ib)(a^*-ib^*)+(a-ib)(a^*+ib^*)\right]$$

$$P=\frac{1}{2}\left[aa^*-iab^*+iba^*+bb^*+aa^*+iab^*-iba^*+bb^*\right]$$

$$P=\left[aa^*+bb^*\right]=|a|^2+|b|^2=1$$

c. We operate on the general state with the S_y^2 operator where

$$S_y^2 = \frac{\hbar^2}{4}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{\hbar^2}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Operating on the general state would return $\frac{\hbar^2}{4}$ times the general state. The probability would of course be unity since the states have been normalized.

4. Griffith's 4.31

Construct S_x, S_y, and S_z for a particle of spin 1. For a spin 1 particle, $m_s = \{-1,0,1\}$ so there are three eigenstates and we can label them as $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $X_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; $X_{-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Let's construct S_z.

$$S_z|\psi\rangle = \hbar m_s|\psi\rangle$$
$$S_z|X_1\rangle = \hbar|X_1\rangle$$
$$S_z|X_0\rangle = 0|X_1\rangle$$
$$S_z|X_{-1}\rangle = -\hbar|X_{-1}\rangle$$
$$S_z = \hbar\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Let's construct S_x and S_y, where as usual $S_x = \frac{S_+ + S_-}{2}$ and $S_y = \frac{S_+ - S_-}{2i}$, where S_+ and S_- are the raising and lowering operators respectively.

$$S_+|X_1\rangle = 0$$
$$S_+|X_0\rangle = \hbar\sqrt{s(s+1) - m_s(m_s+1)}|X_1\rangle = \hbar\sqrt{1(1+1) - 0(0+1)}|X_1\rangle = \sqrt{2}\hbar|X_1\rangle$$
$$S_+|X_{-1}\rangle = \hbar\sqrt{1(1+1) - (-1)(-1+1)}|X_0\rangle = \sqrt{2}\hbar|X_0\rangle$$
$$S_-|X_1\rangle = \hbar\sqrt{s(s+1) - m_s(m_s-1)}|X_0\rangle = \hbar\sqrt{1(1+1) - 0(0-1)}|X_0\rangle = \sqrt{2}\hbar|X_0\rangle$$
$$S_-|X_0\rangle = \hbar\sqrt{1(1+1) - (1)(1-1)}|X_{-1}\rangle = \sqrt{2}\hbar|X_{-1}\rangle$$
$$S_-|X_{-1}\rangle = 0$$
$$S_+ = \sqrt{2}\hbar\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \& \quad S_- = \sqrt{2}\hbar\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Therefore S_x and S_y, are respectively $S_x = \frac{\sqrt{2}}{2}\hbar\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $S_y = \frac{\sqrt{2}}{2}i\hbar\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

5. Griffith's 4.33
   a. The Hamiltonian for the system looks like

$$H=-\gamma\vec{B}\cdot\vec{S}=-\frac{\gamma B_0\hbar}{2}\cos(\omega t)\begin{bmatrix}1&0\\0&-1\end{bmatrix}=\begin{bmatrix}-\frac{\gamma B_0\hbar}{2}\cos(\omega t)&0\\0&\frac{\gamma B_0\hbar}{2}\cos(\omega t)\end{bmatrix}$$

   b. We have to solve the time-dependent Schrodinger wave equation.

$$H|\psi\rangle=i\hbar\frac{d}{dt}|\psi\rangle$$

$$\begin{bmatrix}-\frac{\gamma B_0\hbar}{2}\cos(\omega t)&0\\0&\frac{\gamma B_0\hbar}{2}\cos(\omega t)\end{bmatrix}\begin{pmatrix}\psi_+\\\psi_-\end{pmatrix}=i\hbar\begin{pmatrix}\frac{d\psi_+}{dt}\\\frac{d\psi_-}{dt}\end{pmatrix}$$

$$\rightarrow\frac{i\gamma B_0}{2}\cos(\omega t)\psi_+=\frac{d\psi_+}{dt}\rightarrow\int\frac{i\gamma B_0}{2}\cos(\omega t)dt=\int\frac{d\psi_+}{\psi_+}\rightarrow\psi_+=A_+e^{\frac{i\gamma B_0}{2\omega}\sin(\omega t)}$$

$$\rightarrow-\frac{i\gamma B_0}{2}\cos(\omega t)\psi_-=\frac{d\psi_-}{dt}\rightarrow-\int\frac{i\gamma B_0}{2}\cos(\omega t)dt=\int\frac{d\psi_-}{\psi_-}\rightarrow\psi_-=A_-e^{-\frac{i\gamma B_0}{2\omega}\sin(\omega t)}$$

$$\therefore|\psi(t)\rangle=\begin{pmatrix}A_+e^{\frac{i\gamma B_0}{2\omega}\sin(\omega t)}\\A_-e^{-\frac{i\gamma B_0}{2\omega}\sin(\omega t)}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}e^{\frac{i\gamma B_0}{2\omega}\sin(\omega t)}\\\frac{1}{\sqrt{2}}e^{-\frac{i\gamma B_0}{2\omega}\sin(\omega t)}\end{pmatrix}$$

   The normalization constant is determined from the initial conditions:

$$|\psi(0)\rangle=X_+^x=|\rightarrow\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}A_+\\A_-\end{pmatrix}$$

   c. The probability of getting $-\frac{\hbar}{2}$ (a spin down state) if you measure S_x is given by:

$$P=\left|\langle\leftarrow|\psi(t)\rangle\right|^2=\left|\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}e^{\frac{i\gamma B_0}{2\omega}\sin(\omega t)}\\\frac{1}{\sqrt{2}}e^{-\frac{i\gamma B_0}{2\omega}\sin(\omega t)}\end{pmatrix}\right|^2$$

$$P=\frac{1}{4}\left|e^{\frac{i\gamma B_0}{2\omega}\sin(\omega t)}-e^{-\frac{i\gamma B_0}{2\omega}\sin(\omega t)}\right|^2=\frac{1}{4}\left[2i\sin\left(\frac{\gamma B_0}{2\omega}\sin(\omega t)\right)\right]^2$$

$$P=\left[-i\sin\left(\frac{\gamma B_0}{2\omega}\sin(\omega t)\right)\right]\left[i\sin\left(\frac{\gamma B_0}{2\omega}\sin(\omega t)\right)\right]$$

$$P=\sin^2\left(\frac{\gamma B_0}{2\omega}\sin(\omega t)\right)$$

   d. The minimum magnetic field needed to cause a complete spin flip is when the argument of the sine is a maximum, which occurs at $\frac{\pi}{2}$. Thus we have,

$$\frac{\pi}{2}=\frac{\gamma B_0}{2\omega}\rightarrow B_0=\frac{\pi\omega}{\gamma}=\frac{\pi\omega}{\gamma}.$$

6. Griffith's 4.49

Spin state of a electron: $|\psi\rangle = A\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}$.

a. Normalizing we determine A. We have:

$$1 = \langle\psi|\psi\rangle = A^*\begin{pmatrix} 1+2i & 2 \end{pmatrix} A\begin{pmatrix} 1-2i \\ 2 \end{pmatrix} = |A|^2(1-2i+2i+4+4) = 9|A|^2$$

$$A = \frac{1}{3}$$

b. The probabilities of measuring $\pm\frac{\hbar}{2}$ with S_z:

$$P_+ = \left|\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1-2i}{3} \\ \frac{2}{3} \end{pmatrix}\right|^2 = \left|\frac{1-2i}{3}\right|^2 = \frac{1}{9}(1-2i)(1+2i) = \frac{5}{9}$$

$$P_- = \left|\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1-2i}{3} \\ \frac{2}{3} \end{pmatrix}\right|^2 = \left|\frac{2}{3}\right|^2 = \frac{4}{9}$$

$$\langle S_z\rangle = \langle\psi|S_z\psi\rangle = \begin{pmatrix} \frac{1+2i}{3} & \frac{2}{3} \end{pmatrix}\begin{bmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{bmatrix}\begin{pmatrix} \frac{1-2i}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{\hbar}{18}$$

c. The probabilities of measuring $\pm\frac{\hbar}{2}$ with S_x:

$$P_+ = \left|\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1-2i}{3} \\ \frac{2}{3} \end{pmatrix}\right|^2 = \frac{1}{2}\left|\frac{1-2i}{3}+\frac{2}{3}\right|^2 = \frac{1}{2}\left|\frac{3-2i}{3}\right|^2 = \frac{1}{18}(3-2i)(3+2i) = \frac{13}{18}$$

$$P_- = \left|\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{1-2i}{3} \\ \frac{2}{3} \end{pmatrix}\right|^2 = \frac{1}{2}\left|\frac{1-2i}{3}-\frac{2}{3}\right|^2 = \frac{1}{2}\left|\frac{-1-2i}{3}\right|^2 = \frac{1}{18}(-1-2i)(-1+2i) = \frac{5}{18}$$

$$\langle S_x\rangle = \langle\psi|S_x\psi\rangle = \begin{pmatrix} \frac{1+2i}{3} & \frac{2}{3} \end{pmatrix}\begin{bmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{bmatrix}\begin{pmatrix} \frac{1-2i}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{4\hbar}{18} = \frac{2\hbar}{9}$$

d. The probabilities of measuring $\pm\frac{\hbar}{2}$ with S_y:

$$P_+ = \left|\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}^*\begin{pmatrix} \frac{1-2i}{3} \\ \frac{2}{3} \end{pmatrix}\right|^2 = \frac{1}{2}\left|\frac{1-2i}{3}-\frac{2i}{3}\right|^2 = \frac{1}{2}\left|\frac{1-4i}{3}\right|^2 = \frac{1}{18}(1+4i)(1-4i) = \frac{17}{18}$$

$$P_- = \left|\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}^*\begin{pmatrix} \frac{1-2i}{3} \\ \frac{2}{3} \end{pmatrix}\right|^2 = \frac{1}{2}\left|\frac{1-2i}{3}+\frac{2i}{3}\right|^2 = \frac{1}{2}\left|\frac{1}{3}\right|^2 = \frac{1}{18}$$

$$\langle S_y\rangle = \langle\psi|S_y\psi\rangle = \begin{pmatrix} \frac{1+2i}{3} & \frac{2}{3} \end{pmatrix}\begin{bmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{bmatrix}\begin{pmatrix} \frac{1-2i}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{8\hbar}{18} = \frac{4\hbar}{9}$$