Physics 220
Homework #8
Spring 2017
Due Wednesday, 5/31/17

1. Griffith's 6.2

a. The unperturbed energies of the harmonic oscillator are: $E_n^0 = \left(n+\tfrac{1}{2}\right)\hbar\omega$, where $\omega = \sqrt{\frac{k}{m}}$. Now we let $\omega' = \sqrt{\frac{k'}{m}} = \omega\sqrt{1+\varepsilon}$ for ε a constant. Expanding ε in a power series we have

$$\omega' = \sqrt{\frac{k'}{m}} = \omega\sqrt{1+\varepsilon} = \omega\left(1+\left(\frac{1}{2}\right)\frac{\varepsilon}{1!}+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{\varepsilon^2}{2!}+\ldots\right) = \omega\left(1+\frac{\varepsilon}{2}-\frac{\varepsilon^2}{8}+\ldots\right).$$ Thus the first order correction to the energies are given by

$$E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega' = \left(n+\tfrac{1}{2}\right)\hbar\omega\left(1+\frac{\varepsilon}{2}-\frac{\varepsilon^2}{8}+\ldots\right) \approx \left(n+\tfrac{1}{2}\right)\hbar\omega\left(1+\frac{\varepsilon}{2}\right)$$

$$E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega + \frac{\varepsilon}{2}\left(n+\tfrac{1}{2}\right)\hbar\omega = E_n^0 + E_n^1$$

b. We need to determine $E_n^1 = \langle \psi_n^0 | H' \psi_n^0 \rangle$, so we need to determine the Hamiltonian that describes the perturbation. Starting with the Hamiltonian we have

$$H = H^0 + H' = \frac{p^2}{2m} + \frac{1}{2}m\omega'^2 x^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(1+\varepsilon)x^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2\varepsilon x^2$$

$$\rightarrow H^0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

$$\rightarrow H' = \frac{1}{2}m\omega^2\varepsilon x^2 = \varepsilon V$$

Therefore we need to evaluate $E_n^1 = \langle \psi_n^0 | H' \psi_n^0 \rangle = \varepsilon \langle \psi_n^0 | V \psi_n^0 \rangle$. To do this, we use Griffith's example 2.5 on page 49. The raising and lowering operators are given as

$$a_+ = \frac{1}{\sqrt{2m\hbar\omega}}(-ip + m\omega x)$$

$$a_- = \frac{1}{\sqrt{2m\hbar\omega}}(ip + m\omega x)$$

Adding these two expressions together we can form a representation of the position operator. Squaring this we get $x^2 = \frac{\hbar}{2m\omega}(a_+ + a_-)^2 = \frac{\hbar}{2m\omega}(a_+a_+ + a_+a_- + a_-a_+ + a_-a_-)$.

Thus we get $E_n^1 = \langle \psi_n^0 | H' \psi_n^0 \rangle = \frac{\varepsilon\hbar\omega}{4}\langle \psi_n^0 | (a_+a_+ + a_+a_- + a_-a_+ + a_-a_-)\psi_n^0 \rangle$.

Expanding this we get:

$$E_n^1 = \frac{\varepsilon\hbar\omega}{4}\left[\langle \psi_n^0 | a_+a_+\psi_n^0 \rangle + \langle \psi_n^0 | a_+a_-\psi_n^0 \rangle + \langle \psi_n^0 | a_-a_+\psi_n^0 \rangle + \langle \psi_n^0 | a_-a_-\psi_n^0 \rangle\right].$$

Using equation 2.66 ($a_+|\psi_n^0\rangle = \sqrt{n+1}|\psi_{n+1}^0\rangle$ and $a_-|\psi_n^0\rangle = \sqrt{n}|\psi_{n-1}^0\rangle$) we can evaluate each term in the above expression.

$\langle\psi_n^0|a_+a_+\psi_n^0\rangle = c\langle\psi_n^0|\psi_{n+2}^0\rangle = 0$ with constant c and the fact that the states are orthogonal.

$\langle\psi_n^0|a_+a_-\psi_n^0\rangle = \sqrt{n}\langle\psi_n^0|a_+\psi_{n-1}^0\rangle = \sqrt{n}\sqrt{n}\langle\psi_n^0|\psi_n^0\rangle = n$

$\langle\psi_n^0|a_-a_+\psi_n^0\rangle = \sqrt{n+1}\langle\psi_n^0|a_-\psi_n^0\rangle = \sqrt{n+1}\sqrt{n+1}\langle\psi_n^0|\psi_n^0\rangle = n+1$

$\langle\psi_n^0|a_-a_-\psi_n^0\rangle = c\langle\psi_n^0|\psi_{n-2}^0\rangle = 0$ with constant c and the fact that the states are orthogonal.

Now we can evaluate the first order correction to the energy:

$E_n^1 = \frac{\varepsilon\hbar\omega}{4}\left[0+n+n+1+0\right] = \frac{\varepsilon\hbar\omega}{4}(2n+1) = \frac{\varepsilon\hbar\omega}{2}\left(n+\frac{1}{2}\right)$ which is the result in part a.

2. Griffith's 6.14

The first order corrections to the energies are given by: $E_n^1 = \langle\psi_n^0|H'\psi_n^0\rangle$ where $H' = -\frac{p^4}{8m^3c^2}$. Thus, $E_n^1 = -\frac{1}{8m^3c^2}\langle\psi_n^0|p^4\psi_n^0\rangle$. We'll use the raising and lowering operators defined in chapter 2. Here we need to subtract a_- from a_+ to determine $p = \frac{\sqrt{2m\hbar\omega}}{2}i(a_+ - a_-)$. Squaring this we get:

$$p^2 = \frac{2m\hbar\omega}{4}\left[i(a_+ - a_-)\times -i(a_+ - a_-)\right] = \frac{m\hbar\omega}{2}(a_+a_+ - a_+a_- - a_-a_+ + a_-a_-).$$

Then we square this to get p^4. We have:

$$p^4 = \frac{m^2\hbar^2\omega^2}{4}(a_+a_+ - a_+a_- - a_-a_+ + a_-a_-)^2$$

$$= \frac{m^2\hbar^2\omega^2}{4}\big[a_+a_+a_+a_+ - a_+a_+a_+a_- - a_+a_+a_-a_+ + a_+a_+a_-a_-$$
$$- a_+a_-a_+a_+ + a_+a_-a_+a_- + a_+a_-a_-a_+ - a_+a_-a_-a_-$$
$$- a_-a_+a_+a_+ + a_-a_+a_+a_- + a_-a_+a_-a_+ - a_-a_+a_-a_-$$
$$+ a_-a_-a_+a_+ - a_-a_-a_+a_- - a_-a_-a_-a_+ + a_-a_-a_-a_-\big]$$

This looks intimidating, but most terms will vanish. In fact when I form the inner product to determine the corrections to the energies any term that doesn't have two raising and two lowering operators will yield an inner product between to orthogonal states. Thus those terms will vanish. So I'm going to cancel these immediately before I proceed farther. What we have left to evaluate is:

$$E_n^1 = -\frac{m^2\hbar^2\omega^2}{32m^3c^2}\langle\psi_n^0|\{a_+a_+a_-a_- + a_+a_-a_+a_- + a_+a_-a_-a_+ + a_-a_+a_+a_- + a_-a_+a_-a_+ + a_-a_-a_+a_+\}\psi_n^0\rangle$$

Using the relations for the raising and lowering operators from problem 5:

$$a_+a_+a_-a_-|\psi_n\rangle = a_+a_+a_-\sqrt{n}|\psi_{n-1}\rangle = \sqrt{n}\sqrt{n-1}a_+a_+|\psi_{n-2}\rangle$$
$$= \sqrt{n}(n-1)a_+|\psi_{n-1}\rangle = n(n-1)|\psi_n\rangle$$

$$a_+a_-a_+a_-|\psi_n\rangle = a_+a_-a_+\sqrt{n}|\psi_{n-1}\rangle = na_+a_-|\psi_n\rangle$$
$$= n\sqrt{n}a_+|\psi_{n-1}\rangle = n^2|\psi_n\rangle$$

$$a_+a_-a_-a_+|\psi_n\rangle = a_+a_-a_-\sqrt{n+1}|\psi_{n+1}\rangle = (n+1)a_+a_-|\psi_n\rangle$$
$$= \sqrt{n}(n+1)a_+|\psi_{n-1}\rangle = n(n+1)|\psi_n\rangle$$

$$a_-a_+a_+a_-|\psi_n\rangle = a_-a_+a_+\sqrt{n}|\psi_{n-1}\rangle = na_-a_+|\psi_n\rangle$$
$$= n\sqrt{n+1}a_-|\psi_{n+1}\rangle = n(n+1)|\psi_n\rangle$$
$$a_-a_+a_-a_+|\psi_n\rangle = a_-a_+a_-\sqrt{n+1}|\psi_{n+1}\rangle = (n+1)a_-a_+|\psi_n\rangle$$
$$= (n+1)\sqrt{n+1}a_-|\psi_{n+1}\rangle = (n+1)^2|\psi_n\rangle$$
$$a_-a_-a_+a_+|\psi_n\rangle = a_-a_-a_+\sqrt{n+1}|\psi_{n+1}\rangle = \sqrt{n+1}\sqrt{n+2}a_-a_-|\psi_{n+2}\rangle$$
$$= \sqrt{n+1}(n+2)a_-|\psi_{n+1}\rangle = (n+1)(n+2)|\psi_n\rangle$$

And now we determine the first order corrections to the energies.

$$E_n^1 = -\frac{m^2\hbar^2\omega^2}{32m^3c^2}\left(n(n-1)+n^2+n(n+1)+n(n+1)+(n+1)^2+(n+1)(n+2)\right)\langle\psi_n^0|\psi_n^0\rangle$$
$$= -\frac{\hbar^2\omega^2}{32mc^2}\left[6n^2+6n+3\right]$$
$$\therefore E_n^1 = -\frac{3\hbar^2\omega^2}{32mc^2}\left[2n^2+2n+1\right]$$

.

There is a second way to do this problem. The relativistic corrections are also given by $E_r^{'} = -\frac{1}{8m^3c^2}\left[\left(E_n^0\right)^2 - 2E_n^0\langle\psi_n^0|V\psi_n^0\rangle + \langle\psi_n^0|V^2\psi_n^0\rangle\right]$. From example 2.5 on page 49 of Griffith's text, $\langle V\rangle = \langle\psi_n^0|V\psi_n^0\rangle = \frac{1}{2}(n+\tfrac{1}{2})\hbar\omega = \frac{1}{2}E_n^0$. Thus,

$$E_r^{'} = -\frac{1}{8m^3c^2}\left[\left(E_n^0\right)^2 - 2E_n^0\left(\frac{E_n^0}{2}\right) + \langle\psi_n^0|V^2\psi_n^0\rangle\right] = -\frac{1}{8m^3c^2}\langle\psi_n^0|V^2\psi_n^0\rangle.$$ The potential energy for the harmonic oscillator is given by $V = \frac{m\omega^2}{2}x^2$ and thus $\langle V^4\rangle = \frac{m^2\omega^4}{4}\langle x^4\rangle$

Using the raising and lowering operators defined above, we add a_- and a_+ to determine $x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$. Squaring this we get:

$$x^2 = \frac{\hbar}{2m\omega}\left[(a_+ + a_-)\times(a_+ + a_-)\right] = \frac{\hbar}{2m\omega}\left(a_+a_+ + a_+a_- + a_-a_+ + a_-a_-\right).$$

Then we square this to get x^4. We have:

$$x^4 = \frac{\hbar^2}{4m^2\omega^2}\left(a_+a_+ + a_+a_- + a_-a_+ + a_-a_-\right)^2$$
$$= \frac{\hbar^2}{4m^2\omega^2}[a_+a_+a_+a_+ + a_+a_+a_+a_- + a_+a_+a_-a_+ + a_+a_+a_-a_-$$
$$+ a_+a_-a_+a_+ + a_+a_-a_+a_- + a_+a_-a_-a_+ + a_+a_-a_-a_- \; .$$
$$+ a_-a_+a_+a_+ + a_-a_+a_+a_- + a_-a_+a_-a_+ + a_-a_+a_-a_-$$
$$+ a_-a_-a_+a_+ + a_-a_-a_+a_- + a_-a_-a_-a_+ + a_-a_-a_-a_-]$$

Thus

$$E_n^1 = -\frac{m^2\hbar^2\omega^4}{32m^3c^2}\langle\psi_n^0|\{a_+a_+a_-a_- + a_+a_-a_+a_- + a_+a_-a_-a_+ + a_-a_+a_+a_- + a_-a_+a_-a_+ + a_-a_-a_+a_+\}\psi_n^0\rangle$$

Evaluating this as above we get

$$E_n^1 = -\frac{m^2\hbar^2\omega^2}{32m^3c^2}\left(n(n-1)+n^2+n(n+1)+n(n+1)+(n+1)^2+(n+1)(n+2)\right)\langle\psi_n^0|\psi_n^0\rangle$$

$$= -\frac{\hbar^2\omega^2}{32mc^2}\left[6n^2+6n+3\right]$$

$$\therefore E_n^1 = -\frac{3\hbar^2\omega^2}{32mc^2}\left[2n^2+2n+1\right]$$

3. Griffith's 6.17
To derive equation 6.66 using equations 6.57 (the relativistic correction) and 6.65 (the spin-orbit correction) we look at two cases: $j = l+s = l+\frac{1}{2}$ and $j = l-s = l-\frac{1}{2}$.

Case i: $j = l+\frac{1}{2}$.

The relativistic correction, equation 6.57: $E_r^1 = -\frac{\left(E_n^0\right)^2}{2mc^2}\left(\frac{4n}{l+\frac{1}{2}}-3\right) = -\frac{\left(E_n^0\right)^2}{2mc^2}\left(\frac{4n}{j}-3\right)$.

The spin-orbit correction, equation 6.65:

$$E_{SO}^1 = \frac{\left(E_n^0\right)^2}{mc^2}\left[\frac{n\left[j(j+1)-\frac{1}{2}\left(\frac{1}{2}+1\right)-l(l+1)\right]}{l\left(l+\frac{1}{2}\right)(l+1)}\right] = \frac{n\left(E_n^0\right)^2}{mc^2}\left[\frac{j(j+1)-\left(j-\frac{1}{2}\right)\left(j+\frac{1}{2}\right)-\frac{3}{4}}{\left(j-\frac{1}{2}\right)(j)\left(j+\frac{1}{2}\right)}\right].$$

Evaluating the expression in brackets we have

$$E_{SO}^1 = \frac{n\left(E_n^0\right)^2}{mc^2}\left[\frac{j(j+1)-\left(j-\frac{1}{2}\right)\left(j+\frac{1}{2}\right)-\frac{3}{4}}{\left(j-\frac{1}{2}\right)(j)\left(j+\frac{1}{2}\right)}\right] = \frac{\left(E_n^0\right)^2}{mc^2}\left[\frac{n}{j\left(j+\frac{1}{2}\right)}\right].$$

Adding the relativistic correction and the spin-orbit correction we get

$E_n^1 = -\frac{\left(E_n^0\right)^2}{2mc^2}\left(\frac{4n}{j}-3\right) + \frac{\left(E_n^0\right)^2}{mc^2}\left[\frac{n}{j\left(j+\frac{1}{2}\right)}\right] = \frac{\left(E_n^0\right)^2}{2mc^2}\left[\frac{2n}{j\left(j+\frac{1}{2}\right)}-\frac{4n}{j}+3\right]$. Getting a common denominator for the first two terms we get $E_n^1 = \frac{\left(E_n^0\right)^2}{2mc^2}\left[3-\frac{4n}{j+\frac{1}{2}}\right]$.

Case ii: $j = l-\frac{1}{2}$.

The relativistic correction: $E_r^1 = -\frac{\left(E_n^0\right)^2}{2mc^2}\left(\frac{4n}{l+\frac{1}{2}}-3\right) = -\frac{\left(E_n^0\right)^2}{2mc^2}\left(\frac{4n}{j+1}-3\right)$.

The spin-orbit correction:

$$E_{SO}^1 = \frac{\left(E_n^0\right)^2}{mc^2}\left[\frac{n\left[j(j+1)-\frac{1}{2}\left(\frac{1}{2}+1\right)-l(l+1)\right]}{l\left(l+\frac{1}{2}\right)(l+1)}\right] = \frac{n\left(E_n^0\right)^2}{mc^2}\left[\frac{j(j+1)-\left(j+\frac{1}{2}\right)\left(j+\frac{3}{2}\right)-\frac{3}{4}}{\left(j+\frac{1}{2}\right)(j+1)\left(j+\frac{3}{2}\right)}\right].$$

Evaluating the expression in brackets we have

$$E_{SO}^1 = \frac{n\left(E_n^0\right)^2}{mc^2}\left[\frac{j(j+1)-\left(j-\frac{1}{2}\right)\left(j+\frac{1}{2}\right)-\frac{3}{4}}{\left(j-\frac{1}{2}\right)(j)\left(j+\frac{1}{2}\right)}\right] = -\frac{\left(E_n^0\right)^2}{mc^2}\left[\frac{n}{\left(j+\frac{1}{2}\right)(j+1)}\right].$$

Adding the relativistic correction and the spin-orbit correction we get

$$E_n^1 = -\frac{\left(E_n^0\right)^2}{2mc^2}\left(\frac{4n}{j}-3\right) - \frac{\left(E_n^0\right)^2}{mc^2}\left(\frac{n}{\left(j+\frac{1}{2}\right)(j+1)}\right) = -\frac{\left(E_n^0\right)^2}{2mc^2}\left[\frac{4n}{j+1}+\frac{2n}{\left(j+\frac{1}{2}\right)(j+1)}-3\right].$$

Factoring out a negative sign and getting a common denominator for the first two terms we get $E_n^1 = \frac{\left(E_n^0\right)^2}{2mc^2}\left[3 - \frac{4n}{j+\frac{1}{2}}\right]$.

4. Griffith's 6.18
   Bohr model:

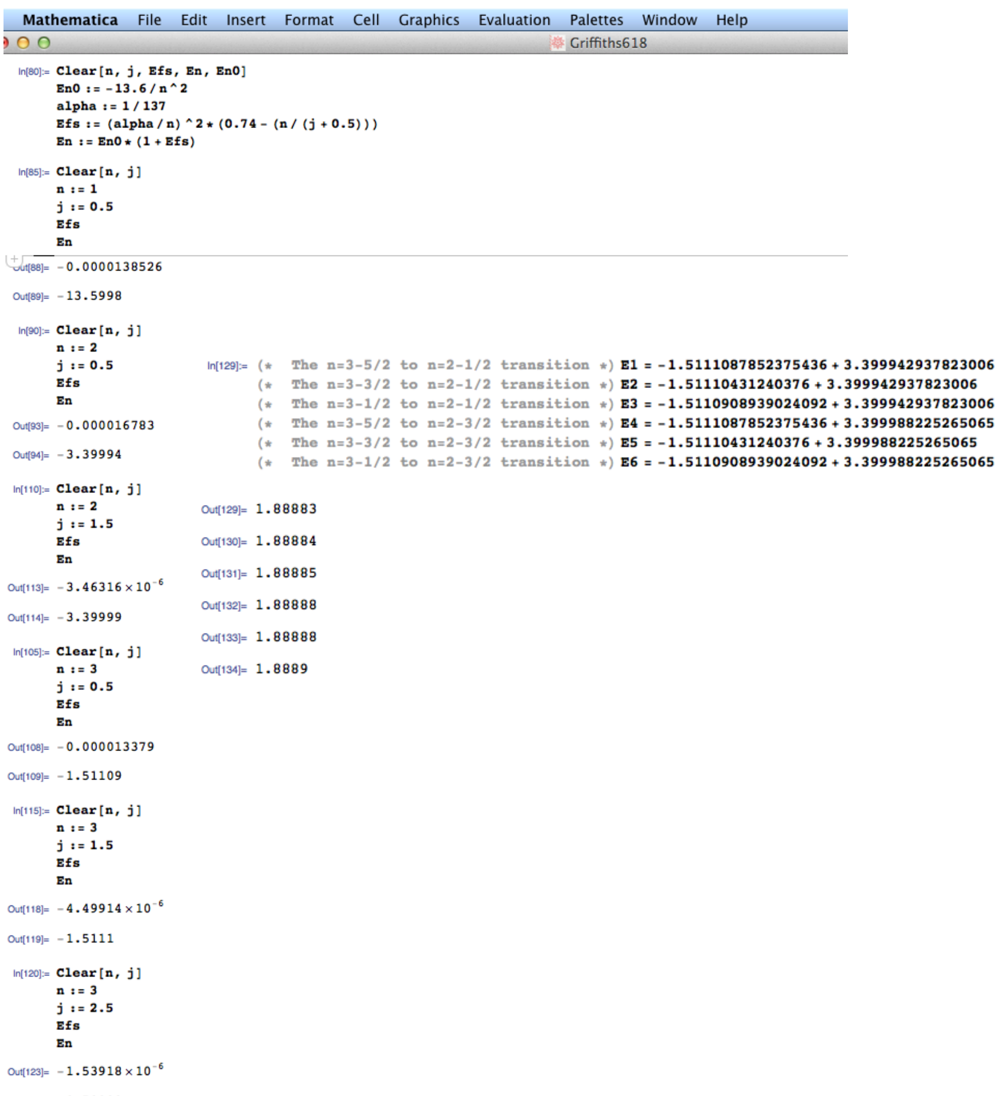
   The Energies of the states are determined on Mathematica and the code is attached below.

$$E_n = -\frac{13.6eV}{n^2}$$

$$\Delta E_{32} = E_3 - E_2 = -13.6eV\left[\frac{1}{3^2} - \frac{1}{2^2}\right] = 1.89eV \times \frac{1.6\times10^{-19}J}{1eV} = 3.022\times10^{-19}J$$

$$\Delta E_{32} = h\upsilon \rightarrow \upsilon = \frac{\Delta E_{32}}{h} = \frac{3.022\times10^{-19}J}{6.63\times10^{-34}Js} = 4.56\times10^{14}s^{-1}$$

$$c = \upsilon\lambda \rightarrow \lambda = \frac{c}{\upsilon} = \frac{3\times10^8\frac{m}{s}}{4.56\times10^{14}s^{-1}} = 6.58\times10^{-7}m = 658nm$$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Mathematica File Edit Insert Format Cell Graphics Evaluation Palettes Window Help
Griffiths618

In[80]:= Clear[n, j, Efs, En, En0]
         En0 := -13.6 / n^2
         alpha := 1 / 137
         Efs := (alpha / n) ^2 * (0.74 - (n / (j + 0.5)))
         En := En0 * (1 + Efs)

In[85]:= Clear[n, j]
         n := 1
         j := 0.5
         Efs
         En
Out[88]= -0.0000138526
Out[89]= -13.5998

In[90]:= Clear[n, j]
         n := 2
         j := 0.5
         Efs
         En
Out[93]= -0.000016783
Out[94]= -3.39994

In[110]:= Clear[n, j]
          n := 2
          j := 1.5
          Efs
          En
Out[113]= -3.46316 × 10^-6
Out[114]= -3.39999

In[105]:= Clear[n, j]
          n := 3
          j := 0.5
          Efs
          En
Out[108]= -0.000013379
Out[109]= -1.51109

In[115]:= Clear[n, j]
          n := 3
          j := 1.5
          Efs
          En
Out[118]= -4.49914 × 10^-6
Out[119]= -1.5111

In[120]:= Clear[n, j]
          n := 3
          j := 2.5
          Efs
          En
Out[123]= -1.53918 × 10^-6
Out[124]= -1.51111

In[129]:= (* The n=3-5/2 to n=2-1/2 transition *) E1 = -1.5111087852375436 + 3.399942937823006
          (* The n=3-3/2 to n=2-1/2 transition *) E2 = -1.51110431240376 + 3.399942937823006
          (* The n=3-1/2 to n=2-1/2 transition *) E3 = -1.5110908939024092 + 3.399942937823006
          (* The n=3-5/2 to n=2-3/2 transition *) E4 = -1.5111087852375436 + 3.399988225265065
          (* The n=3-3/2 to n=2-3/2 transition *) E5 = -1.51110431240376 + 3.399988225265065
          (* The n=3-1/2 to n=2-3/2 transition *) E6 = -1.5110908939024092 + 3.399988225265065

Out[129]= 1.88883
Out[130]= 1.88884
Out[131]= 1.88885
Out[132]= 1.88888
Out[133]= 1.88888
Out[134]= 1.8889
```