Physics 220

In-Class Exam #1

April 24, 2015

Name___________________________________

Please read and follow these instructions carefully:

- Read all problems carefully before attempting to solve them.
- Your work must be legible, and the organization clear.
- You must show all work, including correct vector notation.
- You will not receive full credit for correct answers without adequate explanations.
- You will not receive full credit if incorrect work or explanations are mixed in with correct work. So erase or cross out anything you don't want graded.
- Make explanations complete but brief. Do not write a lot of prose.
- Include diagrams.
- Show what goes into a calculation, not just the final number. For example $|\vec{p}| \approx m|\vec{v}| = (5kg)\times(2\frac{m}{s}) = 10\frac{kg\cdot m}{s}$
- Give standard SI units with your results unless specifically asked for a certain unit.
- Unless specifically asked to derive a result, you may start with the formulas given on the formula sheet including equations corresponding to the fundamental concepts.
- Go for partial credit. If you cannot do some portion of a problem, invent a symbol and/or value for the quantity you can't calculate (explain that you are doing this), and use it to do the rest of the problem.
- Each free-response part is worth 10 points

Problem #1	/30
Problem #2	/30
Problem #3	/30
Total	/90

I affirm that I have carried out my academic endeavors with full academic honesty.

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1. Suppose that you have two light sources that are incident on the same metal surface. One emits photons with wavelength λ_1 while the other emits photons with $\lambda_2 = \frac{\lambda_1}{2}$. The first light source produces photoelectrons with kinetic energy $1eV$ while the second source produces photoelectrons with energy $4eV$.

   a. What is the work function of the metal?

   The kinetic energy of the ejected electrons is

$$K_1 = \frac{hc}{\lambda_1} - \phi$$

$$K_2 = \frac{hc}{\lambda_2} - \phi = \frac{2hc}{\lambda_1} - \phi$$

   Subtracting the two expressions we can determine λ_1 and then the work function of the metal. We have:

$$K_2 - K_1 = \frac{2hc}{\lambda_1} - \phi - \left(\frac{hc}{\lambda_1} - \phi\right) = \frac{hc}{\lambda_1} \rightarrow \lambda_1 = \frac{hc}{K_2 - K_1}$$

$$\lambda_1 = \frac{hc}{K_2 - K_1} = \frac{6.63\times10^{-34}\,Js \times 3\times10^{8}\,\tfrac{m}{s} \times \dfrac{1eV}{1.6\times10^{-19}\,J}}{4eV - 1eV} = 4.14\times10^{-7}\,m$$

   and the work function is

$$K_1 = \frac{hc}{\lambda_1} - \phi \rightarrow \phi = \frac{hc}{\lambda_1} - K_1 = \frac{6.63\times10^{-34}\,Js \times 3\times10^{8}\,\tfrac{m}{s} \times \dfrac{1eV}{1.6\times10^{-19}\,J}}{4.14\times10^{-7}\,m} - 1eV = 2eV$$

b. What is the maximum wavelength of light that could be used in this photoelectric effect experiment?

$$0 = K = \frac{hc}{\lambda} - \phi \rightarrow \lambda_{\max} = \frac{hc}{\phi} = \frac{6.63\times10^{-34}\,Js \times 3\times10^{8}\,\frac{m}{s}}{2eV \times \frac{1.6\times10^{-19}\,J}{1eV}} = 6.22\times10^{-7}\,m = 622nm$$

c. Suppose that the first light source has a power of $1mW = 1\times10^{-3}W$ while the second is $3mW = 3\times10^{-3}W$. What is the ratio of the photocurrent produced by the first light source to the second light source assuming 100% efficiency in each case?

The photocurrent is proportional to the number of electrons ejected per second. To determine the number of electrons ejected (assuming 100% efficiency) is equal to the number of incident photons per second.

$$P = \left(\frac{N_{photons}}{s}\right) E_{photon}$$

$$\frac{N_{photons,1}}{s} = \frac{P_1}{E_1} = \frac{P_1\lambda_1}{hc}$$

$$\frac{N_{photons,2}}{s} = \frac{P_2}{E_2} = \frac{P_2\lambda_2}{hc} = \frac{P_2\lambda_1}{2hc}$$

$$\therefore \frac{\frac{N_{photons,1}}{s}}{\frac{N_{photons,2}}{s}} = \frac{\frac{P_1\lambda_1}{hc}}{\frac{P_2\lambda_1}{2hc}} = \frac{2P_1}{P_1} = \frac{2\times1\times10^{-3}W}{3\times10^{-3}W} = 0.66$$

2. Suppose that a proton is confined in an infinite square well of width $10\,fm = 10\times10^{-15}\,m$. The nuclear potential that binds protons and neutrons is often approximated as an infinite well.

   a. What is the wavelength of the emitted photon if the proton makes a transition from the first excited state to the ground state?

   The energies of the infinite square well are $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$. The energy of the emitted photon is the difference between the energy of the first excited state $n=2$ and the ground state $n=1$.

$$\Delta E = \frac{hc}{\lambda} = E_2 - E_1 = \left(2^2 - 1^2\right)\frac{\pi^2\hbar^2}{2ma^2} = \frac{3\pi^2\hbar^2}{2ma^2} = \frac{3\pi^2\left(\frac{6.63\times10^{-34}\,Js}{2\pi}\right)^2}{2\times1.67\times10^{-27}\,kg\left(10\times10^{-15}\,m\right)^2}$$

$$\frac{hc}{\lambda} = \frac{3\pi^2\hbar^2}{2ma^2} \to \lambda = \frac{2ma^2hc}{3\pi^2\hbar^2}$$

$$\therefore \lambda = \frac{2\times1.67\times10^{-27}\,kg\left(10\times10^{-15}\,m\right)^2\times6.63\times10^{-34}\,Js\times3\times10^8\,\tfrac{m}{s}}{3\pi^2\left(\frac{6.63\times10^{-34}\,Js}{2\pi}\right)^2} = 2.02\times10^{-13}\,m$$

   b. Suppose that photons from this transition are used in a Compton effect experiment. If the photons are completely backscattered from their interaction with a stationary electron, what is the kinetic energy of the electron?

$$K = E - E' = \frac{hc}{\lambda} - \frac{hc}{\lambda'} = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)$$

$$K = 6.63\times10^{-34}\,Js\times3\times10^8\,\tfrac{m}{s}\left(\frac{1}{2.02\times10^{-13}\,m} - \frac{1}{5.05\times10^{-12}\,m}\right)$$

$$\therefore K = 9.476\times10^{-13}\,J = 5.923\,MeV$$

   where the scattered wavelength was calculated from:

$$\lambda' = \lambda + \frac{h}{mc}\left(1-\cos\phi\right) = \lambda + \frac{2h}{mc}$$

$$\lambda' = 2.02\times10^{-13}\,m + \frac{2\times6.63\times10^{-34}\,Js}{9.11\times10^{-31}\,kg\times3\times10^8\,\tfrac{m}{s}} = 5.05\times10^{-12}\,m$$

c. What is the speed of the electron after the interaction expressed as a fraction of the speed of light?

$$K = (\gamma - 1)mc^2 \rightarrow \gamma = 1 + \frac{K}{mc^2} = \sqrt{1 - \frac{v^2}{c^2}} \rightarrow v = \sqrt{1 - \frac{1}{\gamma^2}}c$$

$$\gamma = 1 + \frac{5.923MeV}{\left(0.511\frac{MeV}{c^2}\right)c^2} = 12.591$$

$$\therefore v = \sqrt{1 - \frac{1}{\gamma^2}}c = \sqrt{1 - \frac{1}{(12.591)^2}}c = 0.997c$$

3. When we examined the infinite square well in class, the well was positioned so that the left edge of the well was at $x=0$ and the right edge was at $x=a$. Of course you can put the well anywhere and make it any size you'd like. Consider the "centered" infinite square well with its left edge at $x=-a$ and its right edge at $x=a$. The potential function is given as shown below.

$$V(x)=\begin{cases} \infty \text{ for } x<-a \\ 0 \text{ for } -a\le x\le a \\ \infty \text{ for } x>a \end{cases}$$

a. What are the normalized solutions to the time independent Schrodinger wave equation in each region? (Hint: you may need the following integrals: $\int_{-a}^{a}\sin^2(qx)dx = a-\frac{\sin(2aq)}{2q}$ and $\int_{-a}^{a}\cos^2(qx)dx = a+\frac{\sin(2aq)}{2q}$ and further you may want to define variables $j=2n$ for j even and $j=2n-1$ for j odd to make calculations easier.)

In the well the potential is zero, so Schrodinger's equation looks like:

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}=E\psi \rightarrow \frac{d^2\psi}{dx^2}=-\frac{2mE}{\hbar^2}\psi=-k^2\psi \rightarrow k=\frac{\sqrt{2mE}}{\hbar}$. The solutions are linear combinations of sines and cosines. So the wave function looks like

$\psi = A\sin(kx)+B\cos(kx)$ and we apply the boundary conditions that $\psi(-a)=0$ and $\psi(a)=0$.

At the left boundary: $\psi(-a)=0=-A\sin(ka)+B\cos(ka)$.

At the right boundary $\psi(a)=0=A\sin(ka)+B\cos(ka)$.

These cannot both be true at the same time, so we have to sets of solutions and we'll call then Odd and Even.

The Even Solutions:

Subtracting the results we get $0=2A\sin(ka)\rightarrow ka=n\pi \rightarrow k=\frac{n\pi}{a}$. The wave function is $\psi = A\sin(kx)=A\sin(\frac{n\pi}{a}x)=A\sin(\frac{2n\pi}{2a}x)=A\sin(\frac{j\pi}{2a}x)$ for j even.

The Odd Solutions:

Adding the results we get $0 = 2B\cos(ka) \rightarrow ka = \left(n-\tfrac{1}{2}\right)\pi \rightarrow k = \dfrac{\left(n-\frac{1}{2}\right)\pi}{a}$. The wave function is

$$\psi = B\cos(kx) = A\cos\left(\frac{\left(n-\frac{1}{2}\right)\pi}{a}x\right) = A\cos\left(\frac{2\left(n-\frac{1}{2}\right)\pi}{2a}x\right) = A\cos\left(\frac{(2n-1)\pi}{2a}\right)$$

$$\psi = A\cos\left(\frac{j\pi}{2a}x\right)$$

for j odd.

To normalize the solutions we apply the normalization condition:

For the Even Solutions:

$1 = \int_{-a}^{a} A^2 \sin^2\left(\frac{j\pi}{2a}x\right)dx = A^2\left[a - \frac{a\sin(j\pi)}{j\pi}\right] \rightarrow A = \frac{1}{\sqrt{a}}$ since for even multiples of π, $\sin(j\pi)$ vanishes.

For the odd Solutions:

$1 = \int_{-a}^{a} B^2 \cos^2\left(\frac{j\pi}{2a}x\right)dx = B^2\left[a + \frac{a\sin(j\pi)}{j\pi}\right] \rightarrow B = \frac{1}{\sqrt{a}}$ since for even multiples of π, $\sin(j\pi)$ vanishes.

Therefore the normalized wave functions of the "centered" infinite square well are:

$\psi_j(x) = \frac{1}{\sqrt{a}}\left[\sin\left(\frac{j\pi}{2a}x\right) + \cos\left(\frac{j\pi}{2a}x\right)\right]$ were the index j can be any integer. If j is odd then we pick out the cosine terms (sine terms vanish) and if j is even we pick out the sine terms (the cosine terms vanish.)

b. What are the allowed energies and how many bound energy states are there?

The energies are obtained from the Schrodinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V\psi=E\psi \rightarrow E\psi=-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}$$

$$\psi_j=\frac{1}{\sqrt{a}}\left(\cos\left(\frac{j\pi}{2a}x\right)+\sin\left(\frac{j\pi}{2a}x\right)\right)$$

$$\frac{d\psi_j}{dx}=\frac{j\pi}{2a\sqrt{a}}\left(-\sin\left(\frac{j\pi}{2a}x\right)+\cos\left(\frac{j\pi}{2a}x\right)\right)$$

$$\frac{d^2\psi}{dx^2}=\frac{j\pi}{2a\sqrt{a}}\frac{d}{dx}\left(-\sin\left(\frac{j\pi}{2a}x\right)+\cos\left(\frac{j\pi}{2a}x\right)\right)=\frac{j^2\pi^2}{4a^2\sqrt{a}}\left(\cos\left(\frac{j\pi}{2a}x\right)+\sin\left(\frac{j\pi}{2a}x\right)\right)$$

$$\therefore \frac{d^2\psi}{dx^2}=\frac{j^2\pi^2}{4a^2}\psi_j$$

So Schrodinger's equation gives:

$$E\psi_j=-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}=-\frac{\hbar^2}{2m}\frac{j^2\pi^2}{4a^2}\psi_j$$

$$\therefore E_j=-\frac{j^2\pi^2\hbar^2}{8ma^2}$$

which look exactly like the infinite square well energies and here too there are an infinite number of bound states.

c. What are the expectation values of the position and momentum for the ground state ψ_1?

The ground state has $n=1$ which corresponds to $j=2n-1=1$. Therefore

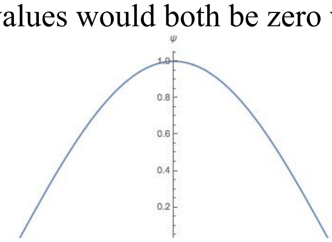
$$\psi_1=\frac{1}{\sqrt{a}}\cos\left(\frac{\pi}{2a}x\right).$$

The expectation value of the position $\langle x\rangle=\int_{-a}^{a}\psi_1^* x\psi_1\,dx=\frac{1}{a}\int_{-a}^{a}x\cos^2\left(\frac{\pi}{2a}x\right)dx=0$.

The expectation value of the momentum

$$\langle p\rangle=\int_{-a}^{a}\psi_1^*\left(-i\hbar\frac{d}{dx}\right)\psi_1\,dx=\frac{i\hbar\pi}{2a^2}\int_{-a}^{a}\cos\left(\frac{\pi}{2a}x\right)\sin\left(\frac{\pi}{2a}x\right)dx=0.$$

If you plot the ground state wave function you'd see that the most likely position would be at $x=0$ and the particle is equally likely to be traveling to the left as the right, so the expectation values would both be zero without doing the integral.



Physics 220 Equations

Useful Integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int e^{ax}\,dx = \frac{e^x}{a}$$

$$\int_{-\infty}^{\infty} e^{ax^2}\,dx = \left(\frac{a}{\pi}\right)^{\frac{1}{4}}$$

$$\int_{-\infty}^{\infty} xe^{ax^2}\,dx = 0$$

$$\int_{-\infty}^{\infty} x^2 e^{ax^2}\,dx = \frac{\sqrt{\pi}}{2a^{\frac{3}{2}}}$$

$$\int_{-\infty}^{\infty} x^2 e^{-\frac{x}{a}}\,dx = \frac{a^3}{4}$$

Constants:

$g = 9.8\,\frac{m}{s^2}$

$G = 6.67\times10^{-11}\,\frac{Nm^2}{kg^2}$

$c = 3\times10^{8}\,\frac{m}{s}$

$\sigma = 5.67\times10^{-8}$

$k_B = 1.38\times10^{-23}\,\frac{J}{K}$

$1eV = 1.6\times10^{-19}\,J$

$h = 6.63\times10^{-34}\,Js;$

$m_e = 9.11\times10^{-31}\,kg = 0.511\,\frac{MeV}{c^2}$

$m_p = 1.67\times10^{-27}\,kg = 938\,\frac{MeV}{c^2}$

$m_n = 1.69\times10^{-27}\,kg = 939\,\frac{MeV}{c^2}$

$m_E = 6\times10^{24}\,kg$

$R_E = 6.4\times10^{6}\,m$

Formulas:

$$c = f\lambda$$

$$E = hf = \frac{hc}{\lambda}$$

$$\frac{dS}{d\lambda} = \frac{2\pi hc^2}{\lambda^5}\left[\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right]$$

$$\frac{dS}{d\lambda} = \frac{2\pi ckT}{\lambda^4}$$

$$\lambda_{\max} = \frac{2.9\times10^{-3}\,m\cdot K}{T}$$

$$S = \sigma T^4$$

$$eV_{stop} = hf - \phi$$

$$\lambda' = \lambda + \frac{h}{mc}(1-\cos\phi)$$

$$\hbar = \frac{h}{2\pi};\ k = \frac{2\pi}{\lambda};\ \omega = 2\pi f$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t} = E\psi$$

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

$$\hat{p} = -i\hbar\frac{\partial}{dx}$$

$$\hat{T} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V$$

$$\hat{x} = x$$

$$\langle O\rangle = \int \psi^*\hat{O}\psi\,dr$$

$$P = \int \psi^*\psi\,dr$$