Physics 220

In-Class Exam #2

May 22, 2015

Name___________________________________

Please read and follow these instructions carefully:

- Read all problems carefully before attempting to solve them.
- Your work must be legible, and the organization clear.
- You must show all work, including correct vector notation.
- You will not receive full credit for correct answers without adequate explanations.
- You will not receive full credit if incorrect work or explanations are mixed in with correct work. So erase or cross out anything you don't want graded.
- Make explanations complete but brief. Do not write a lot of prose.
- Include diagrams.
- Show what goes into a calculation, not just the final number. For example $|\vec{p}| \approx m|\vec{v}| = (5kg)\times(2\,\tfrac{m}{s}) = 10\,\tfrac{kg\cdot m}{s}$
- Give standard SI units with your results unless specifically asked for a certain unit.
- Unless specifically asked to derive a result, you may start with the formulas given on the formula sheet including equations corresponding to the fundamental concepts.
- Go for partial credit. If you cannot do some portion of a problem, invent a symbol and/or value for the quantity you can't calculate (explain that you are doing this), and use it to do the rest of the problem.
- Each free-response part is worth 25 points

Problem #1	/50
Problem #2	/50
Total	/100

I affirm that I have carried out my academic endeavors with full academic honesty.

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1. An electron (mass m and charge e) is confined inside a hollow sphere of radius a and the spherical wall is impenetrable.

(Hints:

- $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\phi^2}$
- Assume a solution $R(r)Y(\theta,\phi)$, let the separation constant be $l(l+1)$, and transform the solution for the radial equation using $R(r) = \frac{u}{r}$.
- The general solution to the equation $-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$ is $u(r) = rR(r) = r\left(AJ_l(kr) + BN_l(kr)\right)$, where $J_l(x) = (-1)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\frac{\sin x}{x}$ are the spherical Bessel functions of order l (and are finite at the origin) and $N_l(x) = (-1)^{l+1}\left(\frac{1}{x}\frac{d}{dx}\right)^l\frac{\cos x}{x}$ are the spherical Neumann functions of order l which are infinite at the origin.
- The following integral may be useful: $\int_0^\infty \frac{\sin^2(nx)}{x^2}dx = \frac{n\pi}{2}$)

a. What is the ground state wave function $\psi_{n,l} = \psi_{0,0}$?

Starting with the SWE:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] + V\psi = E\psi$$

Let $\psi = R(r)Y(\theta,\phi)$ and insert into the SWE:

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial RY}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial RY}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\frac{\partial^2 RY}{\partial\phi^2}\right]-\frac{2m}{\hbar^2}VRY=-\frac{2m}{\hbar^2}ERY$$

$$\left[\frac{Y}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{R}{r^2\sin\theta}\frac{\partial^2 Y}{\partial\phi^2}\right]-\frac{2m}{\hbar^2}VRY=-\frac{2m}{\hbar^2}ERY$$

$$\left[\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{Y\sin\theta}\frac{\partial^2 Y}{\partial\phi^2}\right]-\frac{2mr^2}{\hbar^2}V=-\frac{2mr^2}{\hbar^2}E$$

$$\left[\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}V+\frac{2mr^2}{\hbar^2}E\right]+\left[\frac{1}{Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{Y\sin\theta}\frac{\partial^2 Y}{\partial\phi^2}\right]=0$$

$$\left[\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}V+\frac{2mr^2}{\hbar^2}E\right]=l(l+1)$$

The radial equation: $\left[\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}V+\frac{2mr^2}{\hbar^2}E\right]=l(l+1)$.

Using the hint that $R=\frac{u}{r}$ and substituting this into the radial equation we have

$$\frac{dR}{dr}=\frac{d}{dr}\left(\frac{u}{r}\right)=-\frac{u}{r^2}+\frac{1}{r}\frac{du}{dr}$$

$$r^2\frac{dR}{dr}=-u+r\frac{du}{dr}$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\frac{d}{dr}\left(-u+r\frac{du}{dr}\right)=-\frac{du}{dr}+\frac{du}{dr}+r\frac{d^2u}{dr^2}=r\frac{d^2u}{dr^2}$$

Substituting this into the radial equation we have:

$$\left[\frac{r^2}{u}\frac{d^2u}{dr^2}-\frac{2mr^2}{\hbar^2}V+\frac{2mr^2}{\hbar^2}E\right]=l(l+1)$$

$$\frac{d^2u}{dr^2}+\left[-\frac{2m}{\hbar^2}V+\frac{2m}{\hbar^2}E-\frac{l(l+1)}{r^2}\right]u=0$$

For the ground state $n=0;\ l=0$ so the radial equation reduces to:

$\frac{d^2u}{dr^2}+\left[\frac{2m}{\hbar^2}E\right]u=0$ where the potential $V=0$ inside of the sphere.

$$\frac{d^2u}{dr^2} = -\frac{2mE}{\hbar^2}u \rightarrow \frac{d^2u}{dr^2} = -k^2u$$

$$\rightarrow u = u(r) = rR(r) = r\left(AJ_l(kr) + BN_l(kr)\right)$$

$$R(r) = \frac{u}{r} = AJ_l(kr) + BN_l(kr) = A\frac{\sin(kr)}{kr} - B\frac{\cos(kr)}{kr}$$

Using the fact that as $r \rightarrow 0$ the solutions need to be finite, let $B = 0$. Therefore, $R(r) = A\frac{\sin(kr)}{kr}$. Normalizing the solution we have:

$\int_0^\infty |R(r)|^2\, dr = \frac{A^2}{k^2}\int_0^\infty \frac{\sin^2(kr)}{r^2}dr = \frac{A^2}{k^2}\frac{k\pi}{2} = 1 \rightarrow A = \sqrt{\frac{2k}{\pi}}$. Thus $R(r) = \sqrt{\frac{2k}{\pi}}\frac{\sin(kr)}{kr}$.

Lastly to form the ground state wave function we join the two solutions:

$$\psi_{n,l} = \psi_{0,0} = R(r)Y_0^0\left(\theta,\phi\right) = \sqrt{\frac{2k}{\pi}}\frac{\sin(kr)}{kr}Y_0^0\left(\theta,\phi\right)$$

b. What is the ground state energy?

From part a,

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = Eu \rightarrow \frac{d^2u}{dr^2} = -\frac{2mE}{\hbar^2}u \rightarrow u = \frac{A}{k}\sin(kr) - \frac{B}{k}\cos(kr).$$

$$k^2 = \frac{2mE}{\hbar^2}$$

Applying the condition that at $r = 0$, $u = 0 = A\sin(0) + B\cos(0)$, we have $B = 0$.

Therefore $u = \frac{A}{k}\sin(kr)$ and at $r = a$, $u = 0 = \frac{A}{k}\sin(ka) \rightarrow ka = n\pi$.

Thus $k^2 = \frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{a^2} \rightarrow E = \frac{n^2\hbar^2\pi^2}{2ma^2} = \frac{\hbar^2\pi^2}{2ma^2}$ for $n = 1$ the ground state.

2. A particle of mass m is incident from the left along the x-axis and strikes a delta-function potential barrier $V(x) = V_0\delta(x)$ where is a constant V_0.

   a. When the particle obeys the Schrödinger equation the wave function $\psi(x)$ to the right of the barrier (for $x > 0$) is connected at with the wave function $\psi(x)$ to the left of the barrier (for $x < 0$). Show that the left and right wave functions satisfy the following relation $\lim_{\varepsilon \to 0}\left[\left(\frac{d\psi(x)}{dx}\right)_{x=-\varepsilon} - \left(\frac{d\psi(x)}{dx}\right)_{x=+\varepsilon}\right] = 2q^2\psi(0)$ and determine the value of q^2.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

$$\lim_{\varepsilon \to 0}\left[\int_{-\varepsilon}^{\varepsilon}\frac{d^2\psi}{dx^2}dx\right] - \frac{2m}{\hbar^2}\lim_{\varepsilon \to 0}\left[\int_{-\varepsilon}^{\varepsilon}V\psi\,dx\right] = -\frac{2m}{\hbar^2}\lim_{\varepsilon \to 0}\left[\int_{-\varepsilon}^{\varepsilon}E\psi\,dx\right]$$

$$\lim_{\varepsilon \to 0}\left[\left.\frac{d\psi}{dx}\right|_{-\varepsilon}^{\varepsilon}\right] - \frac{2m}{\hbar^2}\lim_{\varepsilon \to 0}\left[\int_{-\varepsilon}^{\varepsilon}V_0\delta(x)\psi\,dx\right] = -\frac{2mE}{\hbar^2}\lim_{\varepsilon \to 0}\left[\int_{-\varepsilon}^{\varepsilon}\psi\,dx\right] = 0$$

$$\lim_{\varepsilon \to 0}\left[\left(\frac{d\psi(x)}{dx}\right)_{x=-\varepsilon} - \left(\frac{d\psi(x)}{dx}\right)_{x=+\varepsilon}\right] = \frac{2mV_0}{\hbar^2}\psi(0) = 2q^2\psi(0)$$

$$\therefore q^2 = \frac{mV_0}{\hbar^2}$$

b. Suppose that the wave function of the particle having energy E is given by e^{ikx} where $k=\sqrt{\frac{2mE}{\hbar^2}}$. Determine the reflection and transmission coefficients (R and T) as functions of q and k. Explicitly determine R and T and show that $R+T=1$. Do not use $T=1-R$ or $R=1-T$.

$x<0, E>0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}=E\psi \rightarrow \frac{d^2\psi}{dx^2}=-\frac{2mE}{\hbar^2}\psi=-k^2\psi \rightarrow \psi=Ae^{ikx}+Be^{-ikx}$$

$x>0, E>0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}=E\psi \rightarrow \frac{d^2\psi}{dx^2}=-\frac{2mE}{\hbar^2}\psi=-k^2\psi \rightarrow \psi=Ce^{ikx}+De^{-ikx}$$

Here, $D=0$ as there is nothing to reflect the waves back as $x\rightarrow\infty$. Now we apply the boundary conditions that the wave function has to be continuous at $x=0$ which gives $\psi(0)=A+B=C$. Now using the results from part a for the discontinuous derivative:

$$\left(\frac{d\psi(x)}{dx}\right)_{x=-0}=ikA-ikB$$

$$\left(\frac{d\psi(x)}{dx}\right)_{x=+0}=ikC$$

$$\lim_{\varepsilon\rightarrow0}\left[\left(\frac{d\psi(x)}{dx}\right)_{x=-\varepsilon}-\left(\frac{d\psi(x)}{dx}\right)_{x=+\varepsilon}\right]=ikC-(ikA-ikB)=2q^2\psi(0)=2q^2(A+B)$$

$$\therefore ik(A-B-C)=ik(A-B-(A+B))=2q^2(A+B)$$

$$-2ikB=2q^2A+2q^2B\rightarrow B=-\frac{q^2}{ik+q^2}A$$

$$R=\frac{B^*}{A^*}\frac{B}{A}=\left(-\frac{q^2}{-ik+q^2}\right)\left(-\frac{q^2}{ik+q^2}\right)=\frac{q^4}{k^2+q^4}$$

Now to calculate the transmission coefficient:

$$\lim_{\varepsilon\rightarrow0}\left[\left(\frac{d\psi(x)}{dx}\right)_{x=-\varepsilon}-\left(\frac{d\psi(x)}{dx}\right)_{x=+\varepsilon}\right]=ikC-(ikA-ikB)=2q^2\psi(0)=2q^2(C)$$

$$\therefore ik(A-(C-A)-C)=ik(2A-2C)=2q^2(C)$$

$$2ikA-2ikC=2q^2C\rightarrow C=\frac{ik}{ik+q^2}A$$

$$T=\frac{C^*}{A^*}\frac{C}{A}=\left(\frac{-ik}{-ik+q^2}\right)\left(\frac{ik}{ik+q^2}\right)=\frac{k^2}{k^2+q^4}\rightarrow T+R=\frac{k^2}{k^2+q^4}+\frac{q^4}{k^2+q^4}=1$$

Physics 220 Equations

Useful Integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int_0^{\infty} \frac{\sin^2(mx)}{x^2}\,dx = \frac{m\pi}{2}$$

$$\int e^{ax}\,dx = \frac{e^x}{a}$$

$$\int_{-\infty}^{\infty} e^{ax^2}\,dx = \left(\frac{a}{\pi}\right)^{\frac{1}{4}}$$

$$\int_{-\infty}^{\infty} x e^{ax^2}\,dx = 0$$

$$\int_{-\infty}^{\infty} x^2 e^{ax^2}\,dx = \frac{\sqrt{\pi}}{2a^{\frac{3}{2}}}$$

$$\int_{-\infty}^{\infty} x^2 e^{-\frac{x}{a}}\,dx = \frac{a^3}{4}$$

Constants:

$g = 9.8\,\frac{m}{s^2}$

$G = 6.67\times 10^{-11}\,\frac{Nm^2}{kg^2}$

$c = 3\times 10^8\,\frac{m}{s}$

$\sigma = 5.67\times 10^{-8}$

$k_B = 1.38\times 10^{-23}\,\frac{J}{K}$

$1eV = 1.6\times 10^{-19}\,J$

$h = 6.63\times 10^{-34}\,Js;$

$m_e = 9.11\times 10^{-31}\,kg = 0.511\,\frac{MeV}{c^2}$

$m_p = 1.67\times 10^{-27}\,kg = 938\,\frac{MeV}{c^2}$

$m_n = 1.69\times 10^{-27}\,kg = 939\,\frac{MeV}{c^2}$

$m_E = 6\times 10^{24}\,kg$

$R_E = 6.4\times 10^6\,m$

Formulas :

$$c = f\lambda$$

$$E = hf = \frac{hc}{\lambda}$$

$$\frac{dS}{d\lambda} = \frac{2\pi hc^2}{\lambda^5}\left[\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right]$$

$$\frac{dS}{d\lambda} = \frac{2\pi ckT}{\lambda^4}$$

$$\lambda_{\max} = \frac{2.9\times 10^{-3}\,m\cdot K}{T}$$

$$S = \sigma T^4$$

$$eV_{stop} = hf - \phi$$

$$\lambda' = \lambda + \frac{h}{mc}(1-\cos\phi)$$

$$\hbar = \frac{h}{2\pi};\ k = \frac{2\pi}{\lambda};\ \omega = 2\pi f$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t} = E\psi$$

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

$$\hat{p} = -i\hbar\frac{\partial}{dx}$$

$$\hat{T} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V$$

$$\hat{x} = x$$

$$\langle O\rangle = \int \psi^*\hat{O}\psi\,dr$$

$$P = \int \psi^*\psi\,dr$$

$$\hat{H}|\psi\rangle = E|\psi\rangle$$