Physics 220

Take Home Exam #1

April 24, 2015

Name___________________________________

Please read and follow these instructions carefully:

- Read all problems carefully before attempting to solve them.
- You must show all work and the work must be legible, and the organization clear.
- You will not receive full credit for correct answers without adequate explanations.
- You will not receive full credit if incorrect work or explanations are mixed in with correct work. So erase or cross out anything you don't want graded.
- Make explanations complete but brief. Do not write a lot of prose.
- Include diagrams and show what goes into a calculation, not just the final number. For example $|\vec{p}| \approx m|\vec{v}| = (5kg)\times(2\frac{m}{s}) = 10\frac{kg\cdot m}{s}$.
- Give standard SI units with your results unless specifically asked for a certain unit.
- Unless specifically asked to derive a result, you may start with the formulas given on the formula sheet including equations corresponding to the fundamental concepts.
- Go for partial credit. If you cannot do some portion of a problem, invent a symbol and/or value for the quantity you can't calculate (explain that you are doing this), and use it to do the rest of the problem.
- You may use your textbook, class notes, and/or Mathematica to solve the problems. If you use Mathematica, make sure you show what goes into the calculation, not just "done on Mathematica." Set the entire problem up and then feel free to evaluate the integrals or the like. Print out your results.
- You may not under any circumstances consult any other texts or the Internet for solutions.
- You are not to consult any other student or instructor in the completion of this exam.
- The exam will be collected on Friday, April 24, 2015 before the in-class exam is distributed.
- Each free-response part is worth 10 points

Problem #1	/10
Problem #2	/30
Problem #3	/10
Problem #4	/30
Problem #5	/30
Total	/110

I affirm that I have carried out my academic endeavors with full academic honesty.

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1. When Planck developed the theoretical model to describe the emission spectrum from a heated object he assumed that the energies were quantized and used this idea (and Maxwell-Boltzmann statistics) to develop the final form of the quantum radiation law (the correction to the Rayleigh-Jeans law). Show that $E_{avg} = \frac{\sum_{n=0}^{\infty} N(n)E_n}{\sum_{n=0}^{\infty} N(n)}$, where the number of oscillators given by $N(n) = N_o e^{-\frac{E_N}{kT}}$ becomes the average energy of a classical oscillator ($E_{avg} = kT$) if there is a continuous distribution of energies.

   To evaluate the above expression we change the discrete sum to an integral over the continuous variable n. We have (where the integrals were evaluated on Mathematica)

$$E_{avg} = \frac{\sum_{n=0}^{\infty} N(n)E_n}{\sum_{n=0}^{\infty} N(n)} \rightarrow \left\langle E_{avg} \right\rangle = \frac{\int_0^{\infty} N_o e^{-n\frac{hc}{\lambda kT}} \frac{nhc}{\lambda} dn}{\int_0^{\infty} N_o e^{-n\frac{hc}{\lambda kT}} dn} = \frac{\frac{\lambda k^2 T^2}{hc}}{\frac{\lambda kT}{hc}} \Rightarrow \left\langle E_{avg} \right\rangle = kT$$

   if there's a continuous distribution of oscillator energies.

2. We've shown (as a homework problem) that for thermal radiation, the relationship between the energy per unit volume per unit wavelength ($\frac{du}{d\lambda}$) and the number of photons per volume per unit wavelength ($\frac{dn}{d\lambda}$) is $\frac{du}{d\lambda} = \left(\frac{hc}{\lambda}\right)\frac{dn}{d\lambda}$.

   a. Using this information, show that the total number of photons per volume is given approximately as $n \approx \left(3\times10^{19}\,eV^{-3}m^{-3}\right)\left(kT\right)^3$. (Hint: $\int_0^\infty \frac{x^2}{e^x-1}dx = 2.4$.)

   In order to do this problem, we need to relate $\frac{dS}{d\lambda}$ to $\frac{du}{d\lambda}$. From the definition of intensity and considering radiation emitted rom a surface we can write

   $\frac{ds}{d\lambda} = \frac{c}{4}\frac{du}{d\lambda} \rightarrow \frac{du}{d\lambda} = \frac{4}{c}\frac{ds}{d\lambda} = \frac{4}{c}\left[\frac{2\pi hc^2}{\lambda^5\left(e^{\frac{hc}{\lambda kT}}-1\right)}\right] = \frac{8\pi hc}{\lambda^5\left(e^{\frac{hc}{\lambda kT}}-1\right)}$. What we need to therefore evaluate is $n = \int_0^\infty\left(\frac{dn}{d\lambda}\right)d\lambda$. Using the expression above, we have:

$$n = \int_0^\infty\left(\frac{dn}{d\lambda}\right)d\lambda = \int_0^\infty\frac{\lambda}{hc}\left(\frac{du}{d\lambda}\right)d\lambda = \int_0^\infty\frac{\lambda}{hc}\left(\frac{8\pi hc}{\lambda^5\left(e^{\frac{hc}{\lambda kT}}-1\right)}\right)d\lambda.$$

$$\therefore n = 8\pi\int_0^\infty\frac{d\lambda}{\lambda^4\left(e^{\frac{hc}{\lambda kT}}-1\right)} = \frac{16\pi\left(kT\right)^3}{\left(hc\right)^3} \approx 3\times10^{19}\,eV^{-3}m^{-3}\left(kT\right)^3$$

b. What is the average photon energy?

The average energy is given by

$$\langle E\rangle = \frac{u}{n} = \frac{4S}{cn} \approx \frac{4\sigma T^4}{cn} = \frac{4\sigma T^4}{c(3\times10^{19}\,eV^{-3}m^{-3}(kT)^3)}$$

$$\langle E\rangle \approx \frac{4\sigma kT}{ck^4\left(3\times10^{19}\,eV^{-3}m^{-3}\right)} = \left(\frac{4\sigma}{ck^4\left(3\times10^{19}\,eV^{-3}m^{-3}\right)}\right)kT$$

$$\langle E\rangle \approx \left(\frac{4\times5.76\times10^{-8}\,\frac{J}{sm^2K^4}}{3\times10^8\,\frac{m}{s}\left(1.38\times10^{-23}\,\frac{J}{K}\right)^4\left(3\times10^{19}\,\frac{1}{eV^3m^3}\right)}\times\left(\frac{1.6\times10^{-19}\,J}{1eV}\right)^3\right)kT$$

$$\therefore \langle E\rangle \approx 2.85kT$$

c. Using the result from parts (a) and (b) what would be the density of cosmic background photons and their average energy if the temperature were $T = 2.73K$?

The density (or number per unit volume) of cosmic background photons is given by

$$N = n(kT)^3 = 3\times10^{19}(eVm)^{-3}\times\left[\left(1.38\times10^{-23}\,\tfrac{J}{K}\times\frac{1eV}{1.6\times10^{-19}\,J}\right)\times2.7K\right]^3$$

$$N \approx 4\times10^8\,m^{-3}$$

Their average energy of a photon is

$$\langle E_{avg}\rangle = 2.85kT = 2.85\times\left(1.38\times10^{-23}\,\tfrac{J}{K}\times\frac{1eV}{1.6\times10^{-19}\,J}\right)\times2.7K$$

$$\langle E_{avg}\rangle = 1.06\times10^{-22}\,J = 6.6\times10^{-4}\,eV$$

3. Suppose that the universe maybe modeled as a black body with temperature T and further, that the universe expands by a constant factor of C. Use this to show that the thermal power spectrum ($\frac{dS}{d\lambda}$) at initial temperature T has the same shape but has a new temperature $\frac{T}{C}$. (Hint: What happens to the wavelengths of photons if the universe expands?)

As the universe expands, the wavelength of the photon expands. If the universe expands by a constant factor of C, then the wavelength of the photon expands by C. The wavelength of the photon is now $\lambda' = C\lambda$.

Now we'll use Planck's correction to the Rayleigh-Jean's law.

$$\frac{dS}{d\lambda'} = \frac{dS}{d\lambda}\frac{d\lambda}{d\lambda'} = \frac{1}{C}\frac{dS}{d\lambda} = \frac{1}{C}\left(\frac{2\pi hc^2}{\lambda^5\left(e^{-\frac{hc}{\lambda kT}}-1\right)}\right) = \frac{2\pi hc^2}{C\left(\frac{\lambda'}{C}\right)^5\left(e^{-\frac{Chc}{\lambda' kT}}-1\right)}$$

$$\frac{dS}{d\lambda'} = \frac{2\pi hc^2 C^4}{(\lambda')^5\left(e^{-\frac{hc}{\lambda' kT'}}-1\right)}$$

which has the exact same form as the Planck radiation law, where we've defined a new temperature $T' = \frac{T}{C}$ which is lower by a factor of C.

We need to check that the intensity is still correct. To do this we integrate the radiation spectrum. We get

$$S = \int_0^\infty \frac{dS}{d\lambda'}d\lambda' = 2\pi hc^2 C^4 \int_0^\infty \frac{d\lambda'}{(\lambda')^5\left(e^{-\frac{hc}{\lambda' kT'}}-1\right)} = 2\pi hc^2 C^4\left(\frac{6k^4 (T')^4}{h^4c^4}\right)$$

$$S = 2\pi hc^2 C^4\left(\frac{6k^4 (T')^4}{h^4c^4}\right) = \sigma'(T')^4$$

The blackbody spectrum in the expanded universe is also a blackbody spectrum but at a lower temperature. (The integrals were evaluated on Mathematica.)

4. Consider the wave function for the infinite square well,

$$\Psi(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{\sqrt{2mE}}{\hbar} x\right) e^{-i\frac{E}{\hbar}t} \text{ with limits } 0 \le x \le a.$$

a. Show that the wave function for a particle of mass m in the infinite square well will return to its original form after a quantum revival time of $T = \frac{4ma^2}{\pi\hbar}$. In other words, show that $\Psi(x,t) = \Psi(x,t+T)$. (Hint: You may need the fact that $e^{-i\theta} = e^{-i(\theta+2\pi k)}$ for k an integer.)

Since the wave function is given by $\Psi(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{\sqrt{2mE}}{\hbar} x\right) e^{-i\frac{E}{\hbar}t}$ the wave function at $\Psi(x,t+T)$ is $\Psi(x,t+T) = \sqrt{\frac{2}{a}} \sin\left(\frac{\sqrt{2mE}}{\hbar} x\right) e^{-i\frac{E}{\hbar}(t+T)}$. Equating the two expressions we find

$$\sqrt{\frac{2}{a}} \sin\left(\frac{\sqrt{2mE}}{\hbar} x\right) e^{-i\frac{E}{\hbar}t} = \sqrt{\frac{2}{a}} \sin\left(\frac{\sqrt{2mE}}{\hbar} x\right) e^{-i\frac{E}{\hbar}(t+T)} \rightarrow e^{-i\frac{E}{\hbar}t} = e^{-i\frac{E}{\hbar}(t+T)}$$

$$1 = e^{-i\frac{E}{\hbar}T} = e^{-i\frac{E}{\hbar}\frac{4ma^2}{\pi\hbar}} = e^{-i\frac{\left(\frac{n^2\pi^2\hbar^2}{2ma^2}\right)}{\hbar}\frac{4ma^2}{\pi\hbar}} = e^{-2\pi i n^2}$$

Using the hint, we see that $e^{-i\theta} = e^{-i(\theta+2\pi k)} = e^{-i\theta} e^{-2\pi i k} \rightarrow 1 = e^{-2\pi i k}$. Therefore comparing to the above we have an identity so the quantum reversal time is as given.

b. What is the classical revival time for a non-relativistic particle of mass m in an infinite square well bounded between $0 \le x \le a$?

The classical revival time is given by:

$$E = \frac{p^2}{2m} = \frac{(mv)^2}{2m} = \frac{m\left(\frac{2a}{T}\right)^2}{2} \rightarrow T = \sqrt{\frac{2ma^2}{E}}$$

c. For what energy are these two revival times equal?

$$T = \sqrt{\frac{2ma^2}{E}} = \frac{4ma^2}{\pi\hbar} \rightarrow E = \frac{\pi^2\hbar^2}{8ma^2}$$

5. Suppose that a particle of mass m has energy $E > 0$ and sees the potential $V(x) = -\frac{4\hbar^2 a^2}{m}\left(\frac{1}{e^{ax}+e^{-ax}}\right)^2$ over the region $-\infty < x < \infty$.

   a. Show that the ground state wave function $\psi_0 = \frac{2A}{e^{ax}+e^{-ax}}$ satisfies the time independent Schrödinger wave equation and determine the ground state energy.

   Schrodinger's equation is $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$. If ψ is a solution then it must satisfy the wave equation. Inserting the solution we $\psi_0 = \frac{2A}{e^{ax}+e^{-ax}}$ find:

$$\psi_0 = \frac{2A}{e^{ax}+e^{-ax}}$$

$$\frac{d\psi_0}{dx} = -\frac{2aAe^{ax}\left(e^{ax}-e^{-ax}\right)}{\left(e^{-ax}+e^{ax}\right)^2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} = -\frac{2Aa^2\hbar^2}{2m}\left[\frac{2\left(e^{ax}-e^{-ax}\right)^2}{\left(e^{-ax}+e^{ax}\right)^3} - \frac{\left(e^{-ax}+e^{ax}\right)}{\left(e^{-ax}+e^{ax}\right)^2}\right]$$

$$= -\frac{a^2\hbar^2}{2m}\left(\frac{2A}{e^{-ax}+e^{ax}}\right)\left(\frac{2\left(e^{ax}-e^{-ax}\right)^2}{\left(e^{-ax}+e^{ax}\right)^2} - 1\right)$$

$$\therefore -\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} = -\frac{a^2\hbar^2}{2m}\left(\frac{2\left(e^{ax}-e^{-ax}\right)^2}{\left(e^{-ax}+e^{ax}\right)^2} - 1\right)\psi_0$$

   Inserting this expression and the potential into the SWE we find:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} + V\psi_0 = -\frac{a^2\hbar^2}{2m}\left(\frac{2\left(e^{ax}-e^{-ax}\right)^2}{\left(e^{-ax}+e^{ax}\right)^2} - 1\right)\psi_0 - \frac{4a^2\hbar^2}{2m}\left(\frac{1}{\left(e^{-ax}+e^{ax}\right)^2}\right)\psi_0$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} + V\psi_0 = -\frac{a^2\hbar^2}{2m}\left(\frac{2\left(e^{ax}-e^{-ax}\right)^2}{\left(e^{-ax}+e^{ax}\right)^2} - 1 + \frac{4}{\left(e^{-ax}+e^{ax}\right)^2}\right)\psi_0$$

$$\therefore -\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} + V\psi_0 = -\frac{a^2\hbar^2}{2m}\psi_0 = E\psi_0$$

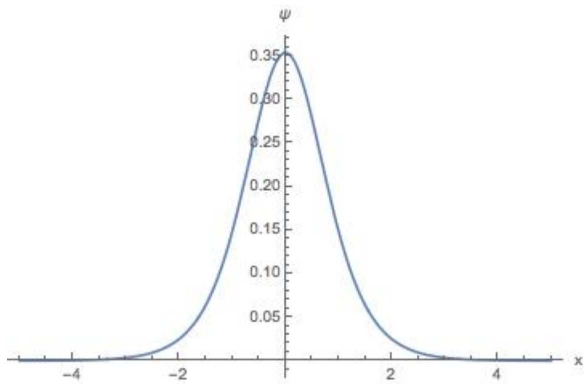
   Therefore ψ_0 satisfies the SWE with ground state energy $E_0 = -\frac{a^2\hbar^2}{2m}$ representing a bound state.

b. Normalize the ground state wave function and plot the solution.

We apply the normalization condition:

$$1=\int_{-\infty}^{\infty}\left|\psi_0\right|^2 dx=\int_{-\infty}^{\infty}\left|\frac{2A}{e^{ax}+e^{-ax}}\right|^2 dx=4A^2\int_{-\infty}^{\infty}\frac{dx}{\left(e^{ax}+e^{-ax}\right)^2}=4A^2\left[\frac{1}{2a}\right]$$

$$\therefore A=\sqrt{\frac{a}{2}}$$



c. What are the reflection and transmission coefficients for this potential?

Since this particle is in a bound state the reflection coefficient $R\sim 1$ while the transmission coefficient $T=1-R\sim 0$. These are approximate since the there is a finite probability of finding the particle outside of the well.

Physics 220 Equations

Useful Integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int e^{ax}\,dx = \frac{e^x}{a}$$

$$\int_{-\infty}^{\infty} e^{ax^2}\,dx = \left(\frac{a}{\pi}\right)^{\frac{1}{4}}$$

$$\int_{-\infty}^{\infty} xe^{ax^2}\,dx = 0$$

$$\int_{-\infty}^{\infty} x^2e^{ax^2}\,dx = \frac{\sqrt{\pi}}{2a^{\frac{3}{2}}}$$

$$\int_{-\infty}^{\infty} x^2e^{-\frac{x}{a}}\,dx = \frac{a^3}{4}$$

Constants:

$g = 9.8\,\frac{m}{s^2}$

$G = 6.67\times10^{-11}\,\frac{Nm^2}{kg^2}$

$c = 3\times10^8\,\frac{m}{s}$

$\sigma = 5.67\times10^{-8}\,\frac{W}{m^2K^4}$

$k_B = 1.38\times10^{-23}\,\frac{J}{K}$

$1eV = 1.6\times10^{-19}\,J$

$h = 6.63\times10^{-34}\,Js;$

$m_e = 9.11\times10^{-31}\,kg = 0.511\,\frac{MeV}{c^2}$

$m_p = 1.67\times10^{-27}\,kg = 938\,\frac{MeV}{c^2}$

$m_n = 1.69\times10^{-27}\,kg = 939\,\frac{MeV}{c^2}$

$m_E = 6\times10^{24}\,kg$

$R_E = 6.4\times10^6\,m$

Formulas:

$c = f\lambda$

$$E = hf = \frac{hc}{\lambda}$$

$$\frac{dS}{d\lambda} = \frac{2\pi hc^2}{\lambda^5}\left[\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right]$$

$$\frac{dS}{d\lambda} = \frac{2\pi ckT}{\lambda^4}$$

$$\lambda_{\max} = \frac{2.9\times10^{-3}\,m\cdot K}{T}$$

$S = \sigma T^4$

$eV_{stop} = hf - \phi$

$$\lambda' = \lambda + \frac{h}{mc}(1-\cos\phi)$$

$$\hbar = \frac{h}{2\pi};\ k = \frac{2\pi}{\lambda};\ \omega = 2\pi f$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t} = E\psi$$

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

$$\hat{p} = -i\hbar\frac{\partial}{dx}$$

$$\hat{T} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V$$

$\hat{x} = x$

$$P = \int \psi^*\psi\,dr$$