Physics 123

Exam #2

November 3, 2006

Problem 1	/32
Problem 2	/48
Multiple-Choice	/20
Total	/100

Free-Response Problems: *Please show all work in order to receive partial credit. If your solutions are illegible no credit will be given. Please use the back of the page if necessary, but number the problem you are working on. Each part of the problem is worth 8 points.*

1. We've seen the hydrostatic condition $\frac{dp}{dz} = -\rho g$. In this question we'll model our atmosphere as being under adiabatic conditions.

   a. Show that there exists an adiabatic relationship between density and pressure and this relationship may be written as $\frac{P}{\rho^{\gamma}} = \text{constant} = \frac{p_1}{\rho_1^{\gamma}}$. (Hint: Start with the definition of density for a gas and write its volume in terms of the number of moles of gas and the molar mass of the gas.)

$$\rho = \frac{M_{gas}}{V} \times \frac{M_{molar,gas}}{M_{molar,gas}} = \frac{n}{V} M_{molar,gas} \rightarrow V = \frac{nM_{molar,gas}}{\rho}$$

$$\therefore PV^{\gamma} = \text{constant} \rightarrow P\left(\frac{nM_{molar,gas}}{\rho}\right)^{\gamma} = \text{constant} \rightarrow \frac{P}{\rho^{\gamma}} = \text{constant} = \frac{P_1}{\rho_1^{\gamma}}$$

   b. From your solution to *part a*, what is the density as a function of pressure? In other words, provide an expression for $\rho(P)$

$$\therefore \frac{P}{\rho^{\gamma}} = \frac{P_1}{\rho_1^{\gamma}} \rightarrow \rho^{\gamma} = \rho_1^{\gamma}\left(\frac{P}{P_1}\right) \rightarrow \rho = \rho_1\left(\frac{P}{P_1}\right)^{\frac{1}{\gamma}}$$

c. Using the hydrostatic condition, what is the pressure *P* as a function of elevation, *z*? (Hint: substitute your expression from *part b* above and integrate from $p = p_1$ when $z = z_1$ to $p = p_2$ when $z = z_2$.)

$$\frac{dP}{dz} = -\rho g \rightarrow \int_{P_1}^{P_2} P^{-\frac{1}{\gamma}} dP = -\int_{z_1}^{z_2} \frac{\rho_1 g}{P_1^{\frac{1}{\gamma}}} dz \rightarrow \left(\frac{\gamma}{\gamma-1}\right)\left(P^{\frac{\gamma-1}{\gamma}}\right)\Big|_{P_1}^{P_2} = -\frac{\rho_1 g}{P_1^{\frac{1}{\gamma}}}(z_2 - z_1)$$

$$\left(\frac{\gamma}{\gamma-1}\right) P_1^{\frac{\gamma-1}{\gamma}} P_1^{\frac{1}{\gamma}} \left(\left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right) = -\frac{\rho_1 g}{P_1^{\frac{1}{\gamma}}}(z_2 - z_1)$$

$$P_2^{\frac{\gamma-1}{\gamma}} = P_1^{\frac{\gamma-1}{\gamma}}\left(1 - \frac{\rho_1 g(\gamma-1)}{\gamma P_1}(z_2 - z_1)\right)$$

$$P = \rho RT \rightarrow \frac{\rho}{P} = \frac{1}{RT}$$

$$\therefore P_2^{\frac{\gamma-1}{\gamma}} = P_1^{\frac{\gamma-1}{\gamma}}\left(1 - \frac{g(\gamma-1)}{\gamma RT_1}(z_2 - z_1)\right) \rightarrow \text{raise to } \frac{\gamma}{\gamma-1} \text{ power.}$$

$$P_2 = P_1\left(1 - \frac{g(\gamma-1)(z_2 - z_1)}{\gamma RT_1}\right)^{\frac{\gamma}{\gamma-1}}$$

d. What are the pressure and density of the atmosphere at an altitude of 1200m if at zero altitude the pressure is $1.013\text{x}10^5$ N/m^2 and the temperature is 15^{o}C, assuming that air is a diatomic gas and that the gas constant for air is R = 287 J/kgK? (If you cannot complete *part c*, you may use the following to evaluate *part d*: $P_2 = P_1\left(1 - \frac{g(\gamma-1)(z_2 - z_1)}{\gamma RT_1}\right)^{\frac{\gamma}{\gamma-1}}$.)

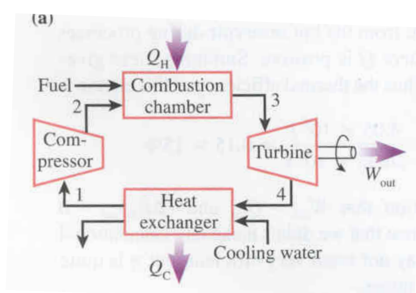
$$P_2 = P_1\left(1 - \frac{g(\gamma-1)(z_2 - z_1)}{\gamma RT_1}\right)^{\frac{\gamma}{\gamma-1}}$$

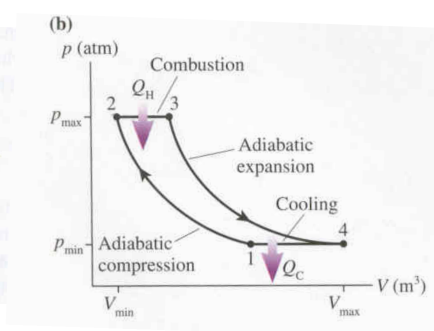
$$= 1.013\times10^5 \tfrac{N}{m^2}\left(1 - \frac{9.8\tfrac{m}{s^2}(1.4-1)(1200m-0)}{1.4\times287\tfrac{J}{kgK}\times288K}\right)^{\frac{1.4}{1.4-1}}$$

$$= 8.733\times10^4 \tfrac{N}{m^2}$$

$$\rho_2 = \frac{RT_1}{P_2} = \frac{287\tfrac{J}{kgK}\times288K}{8.733\times10^4\tfrac{N}{m^2}} = 0.946\tfrac{kg}{m^3}$$

2. The *Brayton Cycle* is the air-standard model of a gas turbine (or jet) engine. Turbine engines are commonly used in power generation plants, ship and aircraft engines, as well as driving a helicopter's rotor blade. The operation of a typical jet engine is shown below. Initially air at atmospheric pressure is rapidly compressed in a compressor. This adiabatic process does work on the gas so that when the air leaves the compressor it is very hot. The gas then flows into a combustion chamber where fuel is continuously admitted and mixes with the hot gas is ignited to heat the gas at constant pressure raising the temperature. The high-pressure gas expands adiabatically, spinning a turbine that does some form of useful work. This drops the temperature and pressure but the gas is still quite hot and the gas is passed through a heat exchanger that transfers heat energy to a cooling fluid. The thermodynamics of the *Brayton Cycle* is also shown below.





a. Suppose that heat $|Q_H|$ is transferred into the gas along the path $2\rightarrow3$ and heat $|Q_C|$ is transferred out of the gas along path $4\rightarrow1$, what is the efficiency of the Brayton Cycle, η_B, in terms of the temperatures T_1, T_2, T_3, and T_4?

$$\eta_B = 1-\frac{|Q_C|}{|Q_H|} = 1-\frac{mc(T_4-T_1)}{mc(T_3-T_2)} \rightarrow \eta_B = 1-\frac{(T_4-T_1)}{(T_3-T_2)}$$

b. The equation you get in *part a* is not very useful unless you know all four temperatures. So let us try something else. Recalling that PV^{γ} is constant in an adiabatic process, solve the ideal gas law for V and show that $P^{1-\gamma}T^{\gamma} = \text{constant}$. This is the pressure temperature relationship for an adiabatic process.

$$PV^{\gamma} = \text{constant} \quad \text{and} \quad PV = nR_0T \rightarrow V = \frac{nR_0T}{P}$$

$$P\left(\frac{nR_0T}{P}\right)^{\gamma} = \frac{P}{P^{\gamma}}(nR_0T)^{\gamma} = \text{constant} \rightarrow P^{1-\gamma}T^{\gamma} = \text{constant}$$

c. Show that your result for *part b* may be written as $P^{\frac{1-\gamma}{\gamma}}T = \text{constant}$.

Raising the result of *part b* to the power of $\frac{1}{\gamma}$ we find

$$\left[P^{1-\gamma}T^{\gamma}\right]^{\frac{1}{\gamma}} = \left[\text{constant}\right]^{\frac{1}{\gamma}} \rightarrow P^{\frac{1-\gamma}{\gamma}}T = \text{constant}$$

d. Process $1 \rightarrow 2$ and $3 \rightarrow 4$ are adiabatic, so using both adiabats express the temperature T_1 in terms of T_2 and the pressure ratio $r_p = \left(\frac{P_{\max}}{P_{\min}}\right)$ and express temperatures T_4 in terms of T_3 and the pressure ratio $r_p = \left(\frac{P_{\max}}{P_{\min}}\right)$.

Using the result of *part c* we have

$$P^{\frac{1-\gamma}{\gamma}}T = \text{constant} \rightarrow P_1^{\frac{1-\gamma}{\gamma}}T_1 = P_2^{\frac{1-\gamma}{\gamma}}T_2 \rightarrow T_1 = T_2\left(\frac{P_2}{P_1}\right)^{\frac{1-\gamma}{\gamma}} = T_2\left(\frac{P_{\max}}{P_{\min}}\right)^{\frac{1-\gamma}{\gamma}} = T_2\left(r_p\right)^{\frac{1-\gamma}{\gamma}}$$

$$P_3^{\frac{1-\gamma}{\gamma}}T_4 = P_3^{\frac{1-\gamma}{\gamma}}T_4 \rightarrow T_4 = T_3\left(\frac{P_3}{P_4}\right)^{\frac{1-\gamma}{\gamma}} = T_3\left(\frac{P_{\max}}{P_{\min}}\right)^{\frac{1-\gamma}{\gamma}} = T_3\left(r_p\right)^{\frac{1-\gamma}{\gamma}}$$

$$\therefore T_1 = T_2\left(r_p\right)^{\frac{1-\gamma}{\gamma}} \quad \text{and} \quad T_4 = T_3\left(r_p\right)^{\frac{1-\gamma}{\gamma}}$$

e. Next express your efficiency to *part a* in terms of the pressure ratio $r_p = \left(\frac{P_{\max}}{P_{\min}}\right)$ which eliminates the temperature dependence and is expressed in terms of something that you can engineer, the pressure ratio of your compressor.

$$\eta_B = 1 - \frac{\left(T_4 - T_1\right)}{\left(T_3 - T_2\right)} = 1 - \frac{\left(T_3\left(r_p\right)^{\frac{1-\gamma}{\gamma}} - T_2\left(r_p\right)^{\frac{1-\gamma}{\gamma}}\right)}{\left(T_3 - T_2\right)} = 1 - \left(r_p\right)^{\frac{1-\gamma}{\gamma}}$$

f. Suppose that your pressure ratio is 10, what is the efficiency of your *Brayton Cycle* engine?

For air, $\gamma = 1.4$ and the efficiency $\eta_B = 1 - \left(r_p\right)^{\frac{1-\gamma}{\gamma}} = 1 - \left(10\right)^{\frac{1-1.4}{1.4}} 1 - 0.518 = 0.482 = 48.2\%$

Short Answer – *Answer each question below in complete sentences. Each short answer question is worth 4 points for a total of 20 points. If you need to do a calculation to support your answer, please use the space below the question.*

1. Suppose that you have a heat engine that can operate in one of two modes. Mode 1 the temperatures of the two reservoirs are T_C = 200K and T_H = 400K, while in mode 2 the temperatures are T_C = 400K and T_H = 600K. Explain how the efficiency of mode 1 compares to mode 2.

   The efficiency of mode 1 is greater than mode 2 as shown below.

$$\varepsilon_1 = 1 - \frac{T_{C1}}{T_{H1}} = 1 - \frac{200K}{400K} = 0.50 = 50\%$$
$$\varepsilon_2 = 1 - \frac{T_{C2}}{T_{H2}} = 1 - \frac{400K}{600K} = 0.33 = 33\%$$

2. One day you look in your refrigerator (with dimensions 1.0m by 0.60m by 0.75m) and find nothing but a dozen eggs (44g each.) How does the weight of the air (density 1.29 kg/m^3) in your refrigerator compare to the weight of the eggs?

   The air weighs more than the eggs as is shown below.

$$W_{eggs} = m_{eggs} g = 12 \times 0.044kg \times 9.8\tfrac{m}{s^2} = 5.17N$$
$$W_{air} = m_{air} g = \rho_{air} V_{air} g = 1.29\tfrac{kg}{m^3}\left(1.0m \times 0.6m \times 0.75m\right) \times 9.8\tfrac{m}{s^2} = 5.69N$$

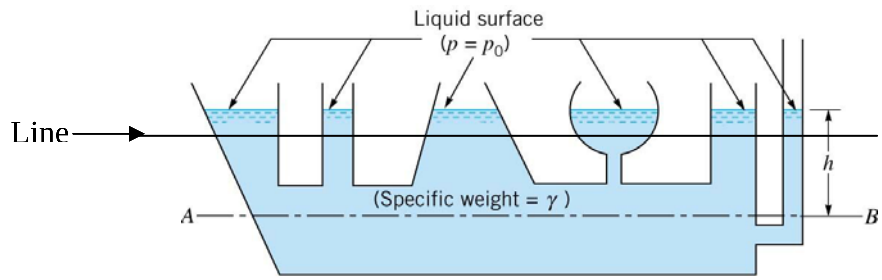
3. One day while swimming below the surface of the ocean you let out a small bubble of air from your mouth. As the bubble rises towards the surface explain how its diameter changes.

   Since the pressure decreases with decreasing depth, the bubble should expand as it rises and thus its radius should increase.

4. Suppose you are drinking a liquid through a straw. Explain why the liquid moves up the straw?

   By sucking on the straw, you create a lower pressure in your mouth than the air pressure pushing down on the surface of the liquid. Therefore there is a pressure differential between your mouth and the liquid surface that forces the liquid up the straw.

5. Given the diagram below, rank the pressures at the line given from largest to smallest. If you cannot rank them explain why. Explain your reasoning in either case.



   They are all at the same pressure since at a given depth the pressure is a constant. The line is at a depth h below the surface, so the pressure $P = P_{air} + \rho g h$.