Physics 123 Homework Solutions

Week #4 Unit T Thermodynamics

T2B.1

According to the ideal gas law, the pressure is given by

$P = \frac{Nk_BT}{V} = \frac{1 molecule \times 1.38\times 10^{-23}\ \frac{J}{K} \times 3K}{1cm^3 \times \frac{1m^3}{(100cm)^3}} = 4.1\times 10^{-17}\,Pa$, which is about 4 orders of magnitude less pressure than the pressure of the best lab vacuum.

T2B.6

The rms velocity of a molecule is given by

$v_{rms} = \sqrt{\frac{3k_BT}{m}} = \sqrt{\frac{3\times 1.38\times 10^{-23}\ \frac{J}{K}\times 295K}{7.3\times 10^{-26}kg}} = 410\ \frac{m}{s}$ where the mass was given by the molar mass of CO_2 (= 12g/mol + 2x16g/mol = 44g/mol) divided by Avogadro's number.

T2B.7

Since Avogadro's number of nucleons has a mass of about 1.0g, Avogadro's number of helium atoms (which contain four nucleons each) should have a mass of 4.0g. Since out sample has a mass of 0.4g, this means that it should contain $1/10^{th}$ of N_A, or $6.02\text{x}10^{22}$ molecules. For monatomic helium, the thermal contained in the gas is $E_{thermal} = \frac{3}{2}Nk_BT = \frac{3}{2}\times 6.02\times 10^{22}\times 1.38\times 10^{-23}\ \frac{J}{K}\times 295K = 368J$. For my mass, which is about 70kg (on a really good day), my speed would have to be $E_{thermal} = \frac{1}{2}mv^2 \rightarrow v = \sqrt{\frac{2E_{thermal}}{m}} = \sqrt{\frac{2\times 368J}{70kg}} = 3.2\ \frac{m}{s}$. This is about 7 mph, which is not that fast, but since a person is very massive, this is a lot of energy contained in the gas.

T2S.2

If we take the ideal gas law and divide by the volume we get $P = \frac{Nk_BT}{V}$. From this expression we see that $P \propto \frac{1}{V}$, which is *Boyle's law* for a fixed N and T.

If we take the ideal gas law and divide by the pressure we get $V = \frac{Nk_BT}{P}$. From this expression we see that $V \propto T$, which is *Gay-Lussac's law* for a fixed N and P.

If we take the ideal gas law and divide by Boltzmann's constant and the temperature we get $\frac{PV}{k_BT} = N$. From this expression we see that $V \propto N$, which for fixed P and T, gives the claim of Avogadro.

T2R.2

a. I'll set up a coordinate system such that the x axis is perpendicular to the end face. Let v_x be the magnitude of a given molecule's velocity and $L = 10m$ is the length of the module parallel to the x axis. Assuming that the collisions with the walls parallel to the x axis do not, on average, change the molecule's x-velocity then the time Δt that it will take the molecule to travel from the end face of interest down the module and back to the end face is $\Delta t = \frac{2L}{|v_x|}$, and this is the time between hits for that molecule. The magnitude is needed in the denominator since the sign of the velocity can be + or -. If there are N molecules in the module that have an average x-velocity of $[v_x]_{avg}$, then the average number of hits that he end face experiences per unit time is $\frac{dN_{hits}}{dt} = N\left(\frac{1hit}{\Delta t_{avg}}\right) = \frac{N|v_x|_{avg}}{2L}$.

The probability that the molecule will go through the hole is when it hit the end face should be the same as the ratio of the hole's area to the face area, or a/A. Any molecule that hits the hole's area will escape to space, which is the absolute value of the rate at which the number of molecules decreases with time.

Therefore $\left|\frac{dN}{dt}\right| = \text{probability of escaping}\frac{\text{dN}_{\text{hits}}}{\text{dt}} = \frac{a}{A}\frac{N}{2L}|v_x|_{avg} = \frac{aN}{2V}|v_x|_{avg}$.

b. Since dN/dt is actually negative, so we can the above equation as $\frac{dN}{dt} = -\frac{aN}{2V}|v_x|_{avg} = -\frac{1}{\tau}N; \quad \tau = \frac{2V}{a|v_x|_{avg}}$. The solutions to this give $N = N_o e^{-\frac{t}{\tau}}$.

c. As a rough guess, I imagine that people would have a difficult time remaining consciousness if the number of molecules in a given amount of inhaled volume drops to roughly *1/3* the number at sea level (see T2A.1). Thus the value of N will drop to N_0/e when $t \sim \tau$, so it is reasonable to assume that people will black out after a time very nearly equal to τ.

d. We'll have to wait until sectionT7 to see this, but for sufficiently narrow distributions of velocities, the square root of the average squared x-velocity is given as $|v_x|_{avg} \approx \sqrt{\frac{k_B T}{m}}$.

e. If we substitute this result into the expression for τ, we have $\tau \approx \frac{2V}{a}\sqrt{\frac{k_B T}{m}}$. If the average mass of an air molecule is m = M/N_A, where M is 29 g/mol (see T2B.3) and the volume of the module is $V = \pi\left(\frac{D}{2}\right)^2 L$ (the module has diameter D.) If the temperature is 295K in the module we find for τ,

$\tau \approx \frac{2\pi\left(\frac{D}{2}\right)^2 L}{a}\sqrt{\frac{M}{N_A k_B T}} = \frac{2\pi(2m)^2 10m}{(0.01m)^2}\sqrt{\frac{0.029\,\frac{kg}{mol}}{8.31\,\frac{J}{kg\times mol}\times 295K}} = 8600s$. This is about 2.5

hours, so the astronauts should have plenty of time to find and patch the leak, I would guess.

T2A.1

a. Let us imagine the parcel to be a flat cylindrical disk with infinitesimal vertical thickness *dz* and horizontal circular top and bottom faces having area *A*. Let the air pressure evaluated at the cylindrical parcel's bottom and top surfaces be *P(z – ½ dz*) and *P(z + ½ dz)* respectively, where *z* specifies the coordinate of the parcel's center. The upward force that air pressure exerts on the bottom of the parcel will then have a magnitude The downward component *AP(z – ½ dz)* and the downward force that air pressure exerts on the top of the parcel will have a magnitude of *AP(z + ½ dz)*. The vertical component of the net force on the particle must be zero, implying that $0 = AP\left(z - \frac{dz}{2}\right) - AP\left(z + \frac{dz}{2}\right) - Nmg \rightarrow P\left(z - \frac{dz}{2}\right) - P\left(z + \frac{dz}{2}\right) = \frac{Nmg}{A}$.

N is the number of molecules in the parcel, *m* is the average mass of a molecule, and *g* is the local gravitational field strength. If we divide both sides by *dz* and the limit that *dz* approaches zero, we find $\lim_{dz \to 0} \frac{P\left(z - \frac{dz}{2}\right) - P\left(z + \frac{dz}{2}\right)}{dz} = -\lim_{dz \to 0} \frac{P\left(z + \frac{dz}{2}\right) - P\left(z - \frac{dz}{2}\right)}{dz} = -\frac{dP}{dz} = \frac{Nmg}{A \times dz} = \frac{Nmg}{V}$.

Note that *N* and *V* both go to zero as *dz* approaches zero, but the ratio *N/V* will remain finite and become the local number density of molecules in this limit. The ideal gas law implies that this ratio $\frac{N}{V} = \frac{P}{k_B T}$ and $-\frac{dP}{dz} = \frac{mg}{k_B T}P$. This is the equation we wanted to show.

b. This is a differential equation with the same form as equations we have encountered in previous courses. The solution is given from $\int_{P_o}^{P} \frac{dP}{P} = -\frac{mg}{k_B T}\int_0^z dz \rightarrow \ln\frac{P}{P_o} = -\frac{mg}{k_B T}z \rightarrow P = P_o e^{-\frac{mg}{k_B T}z}$. To evaluate the pressure at a given altitude, we need to calculate the constants in the exponential. For the molar mass of air, 0.029 kg/mol we have $\frac{mg}{k_B T} = \frac{\left(M/N_A\right)g}{k_B T} = \frac{0.029\,\frac{kg}{mol} \times 9.8\,\frac{m}{s^2}}{6.02 \times 10^{23} \times 1.38 \times 10^{-23}\,\frac{J}{K} \times 273K} = 1.25 \times 10^{-4}\,m^{-1}$ which taking the inverse gives a scale height of 7980m. This is the vertical displacement that causes the pressure to drop by a factor of e and I also assumed that the temperature is constant at this height, but that may not be a good assumption. $P = P_o e^{-\frac{mg}{k_B T}z} = P_o e^{-\frac{8840m}{7980m}} = 0.33P_0$.

Thus at the top of Mount Everest, we have about 1/3 of the pressure found at sea level, or 33kPa.