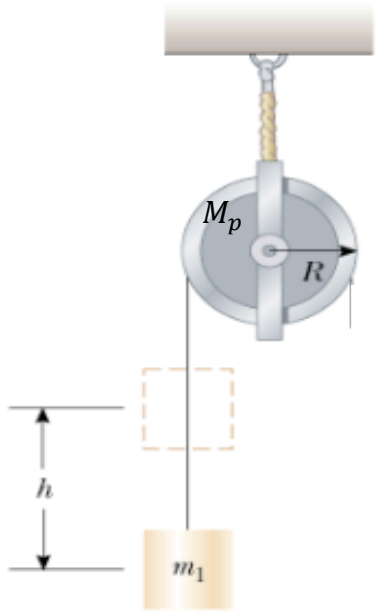
Rotational Dynamics

Example 1:

Consider a pulley of radius $R_P = R$ and mass M_p. A light string is wound around the outer edge of the pulley and a mass m_1 is suspended.

The mass m_1 is released from rest and is allowed to fall through a height of $|\Delta y| = h$.



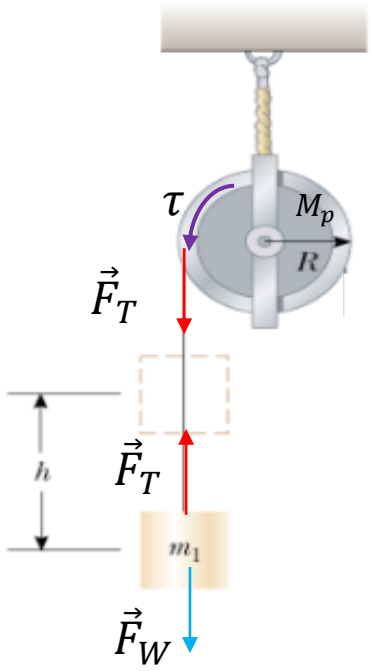
- What is the torque created about the axis of rotation of the pulley due to the falling mass?
- What is the angular acceleration α of the pulley about the axis of rotation due to the falling mass?
- What is the rotational speed ω of the pulley about its center if the mass m_1 has fallen through a height h from rest?
- Through how much angle θ has the pulley spun if m_1 has fallen through a height h from rest?

Rotational Motion

Example 1 Solution:

- Since the mass m_1 falls vertically, a force is exerted on the outer edge of the pulley.
- This force is the tension force that is created in the rope due to the suspended mass falling.
- We'll draw the forces that act on the mass and the pulley.
- The tension force applied at the outer edge of the pulley produces a counterclockwise rotation of the pulley due to a torque created about the center of the pulley (The axis of rotation) and in the last lecture we said that counterclockwise rotations we'll take as a positive.
- The tension force is applied tangentially to the outer edge of the pulley, and this makes the angle between the moment arm (the distance from the axis of rotation about the center of the pulley) and the applied force $\theta = 90^0$.
- The torque produced on the pulley by the tension force is then:

$$\tau = rF_T \sin\theta = RF_T \sin 90 = RF_T$$



Rotational Motion

Example 1 Solution continued:

The external torque causes the pulley to accelerate about the axis of rotation. By Newton's laws of motion, we have:

$$\tau = RF_T = I_P\alpha$$

I_P is the moment of inertia of the pulley and we look up the moment of inertia for a cylinder (or a disk) that is rotate about an axis through its center. Looking up the moment of inertia from our textbook in figure 8.20 on page 210:

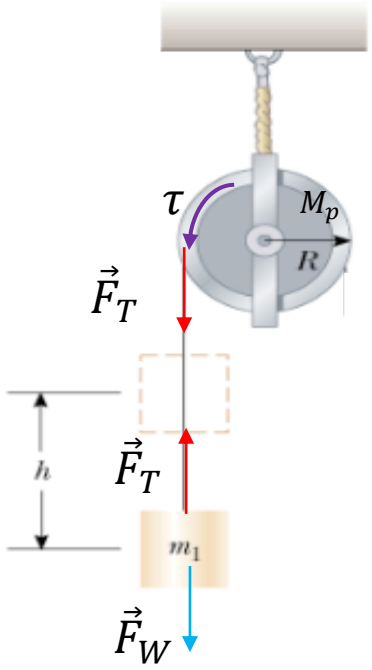
$$I_P = \frac{1}{2}M_P R^2$$

The angular acceleration (α) about the axis of rotation is in the direction of the external torque (counterclockwise about the axis of rotation) and is positive with a magnitude of:

$$\tau = I_P\alpha \rightarrow \alpha = \frac{\tau}{I_P} = \frac{RF_T}{\frac{1}{2}M_P R^2}$$

The one thing we don't know yet is the tension force, F_T.

To determine the tension force, we again look at the other object in the problem and use Newton's laws for a translating mass.



Rotational Motion

Example 1 Solution continued:

- There is nothing else associated with the pulley that will allow me to determine the tension force, F_T. Examining the other object in the system, mass m_1 will give an expression for F_T.

- Taking vertically up as the positive y-direction, we have for the forces that act on mass m_1, the tension force is:

$$F_T - F_W = F_T - m_1 g = m_1 a_y = -m_1 a$$

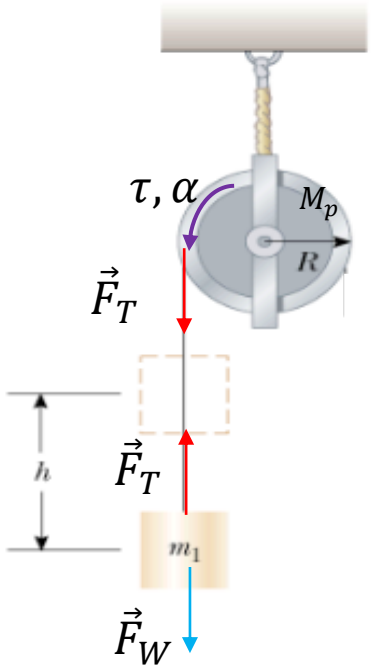
$$\longrightarrow \quad F_T = m_1 g - m_1 a$$

- The vertical acceleration of the mass m_1 is in the negative y-direction and let's call the magnitude simply $|\vec{a}_y| = a$.

- The angular acceleration of the pulley about its axis of rotation is then:

$$\alpha = \frac{\tau}{I_P} = \frac{RF_T}{\frac{1}{2}M_P R^2} = \frac{R(m_1 g - m_1 a)}{\frac{1}{2}M_P R^2}$$

- However, we have to relate the translational acceleration of the falling mass (a) to the angular acceleration of the pulley about the axis of rotation (α).



Rotational Motion

Example 1 Solution continued:

- Since the rope comes off the outer edge of the pulley the tangential acceleration of a piece of rope (a) is related to the angular acceleration (α) of the pulley about the axis of rotation and where you want to evaluate the tangential acceleration, at the outer edge of the pulley, R.

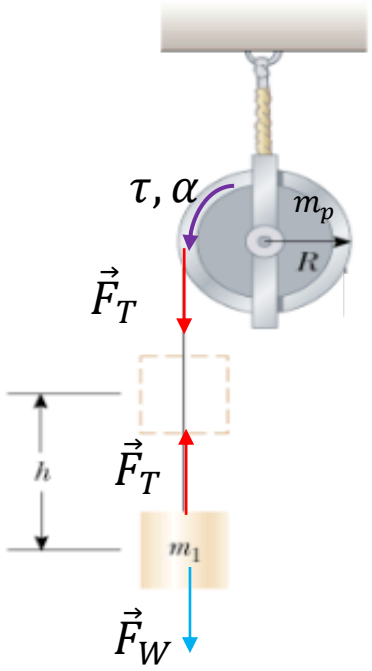
- Thus, we have $a = R\alpha$ and the angular acceleration α is:

$$\alpha = \frac{\tau}{I_P} = \frac{RF_T}{\frac{1}{2}M_P R^2} = \frac{R(m_1 g - m_1 a)}{\frac{1}{2}M_P R^2} = \frac{R(m_1 g - m_1 R\alpha)}{\frac{1}{2}M_P R^2}$$

- Thus, we have $a = R\alpha$ and the angular acceleration is, after some algebra:

$$\boxed{\alpha = \left(\frac{m_1}{\frac{1}{2}m_p + m_1}\right)\frac{g}{R}}$$

This is dimensionally correct and seems reasonable.



$m_p << m_1$

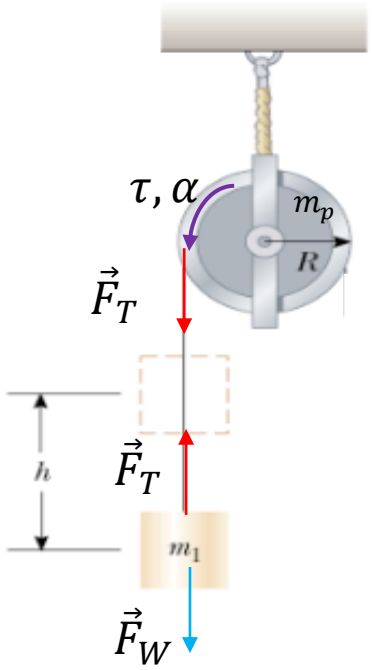
$$\alpha = \left(\frac{m_1}{\frac{1}{2}m_p + m_1}\right)\frac{g}{R} \sim \frac{g}{R} \rightarrow a \sim g$$

$m_p >> m_1$

$$\alpha = \left(\frac{m_1}{\frac{1}{2}m_p + m_1}\right)\frac{g}{R} \sim 0 \rightarrow a \sim 0$$

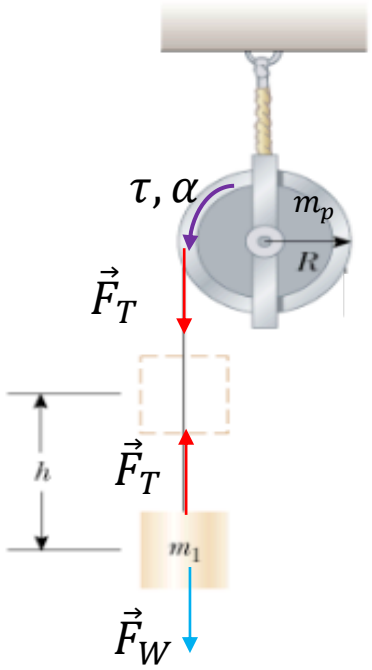
Rotational Motion

Example 1 Solution continued:



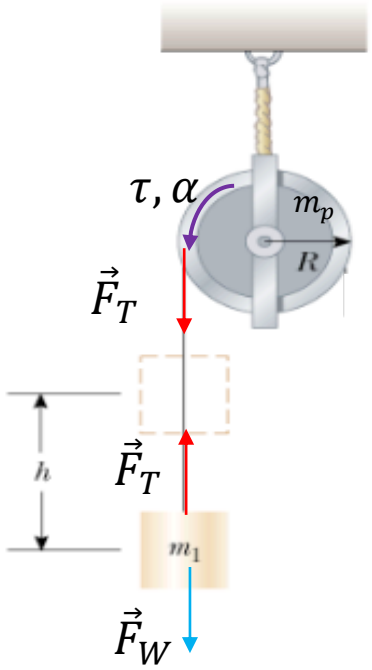
Rotational Motion

Example 1 Solution continued:



Rotational Motion

Example 1 Solution continued:

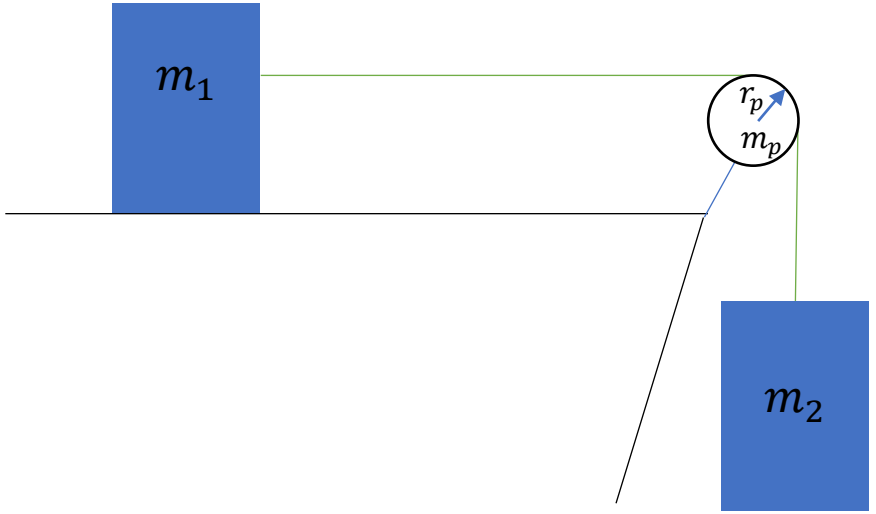


Rotational Dynamics

Example 2:

Suppose a box of mass $m_1 = 2kg$ is connected to a box of mass $m_2 = 1kg$ by a light rope that is passed over a massive pulley ($m_p = 250g$ and $r_p = 2.5cm$). The system is released from rest and m_2 is allowed to fall through a height of $|\Delta y| = 1m$.

- Using forces/torques what is the translational acceleration of the system. ?
- How long does it take for mass m_1 to move a horizontal distance of $1m$ from rest?
- Using this time, what is the translational speed of mass m_1 and the rotational speed of the pulley?
- Through how many radians did the pulley turn? How many revolutions is this?
- Using the distance moved by mass m_1, what is the is the translational speed of mass m_1 and the rotational speed of the pulley?

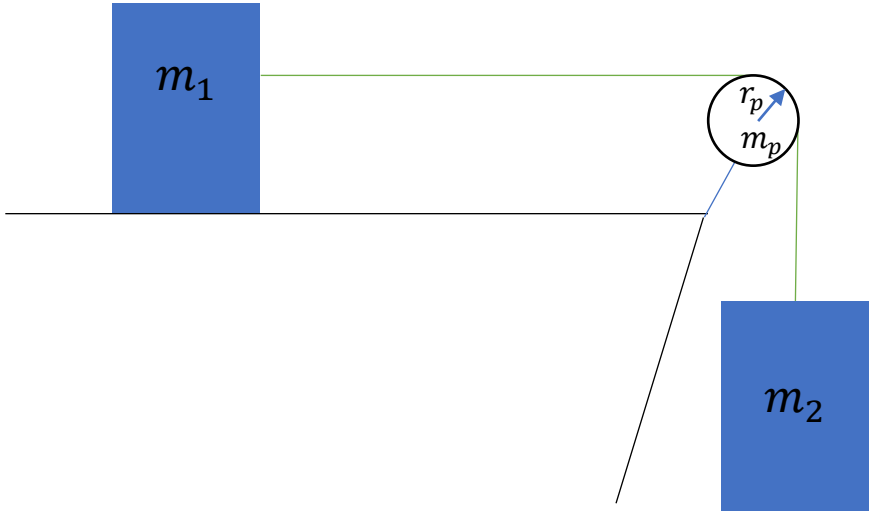


Rotational Dynamics

Example 3:

Suppose a box of mass $m_1 = 2kg$ is connected to a box of mass $m_2 = 1kg$ by a light rope that is passed over a massive pulley ($m_p = 250g$ and $r_p = 2.5cm$). The system is released from rest and m_2 is allowed to fall through a height of $|\Delta y| = 1m$.

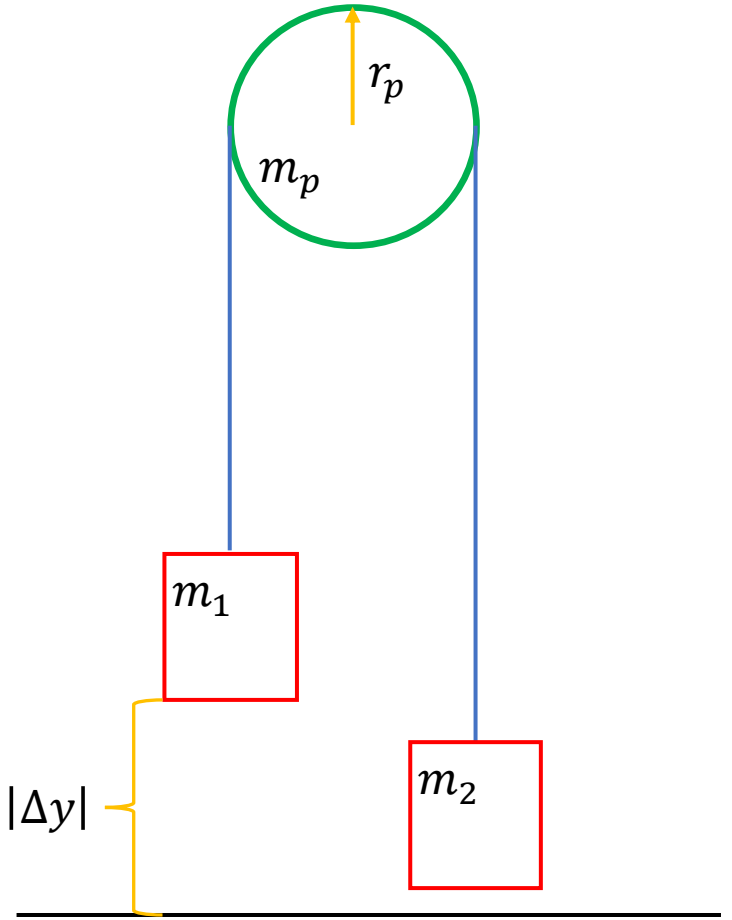
- How much work was done on mass m_1?
- How much work was done on mass m_2?
- How much work did it take to spin the pulley?
- Using the work-kinetic energy theorem, what is the translational speed of mass m_1?
- Using energy ideas what is the translational speed of mass m_1?



Rotational Dynamics

Example 4:

Suppose a box of mass $m_1 = 2kg$ is released from rest and allowed to fall through a height of $|\Delta y|$. Assume m_1 is connected to a mass $m_2 = 1kg$ by a massless rope that passes over a massive pulley with mass $m_p = 0.25kg$ and radius $r_p = 0.1m$. This arrangement of masses is called an Atwood machine.



- Using forces/torques, what is the magnitude of the translational acceleration of the system of masses?
- How does this compare this to the case if the pulley was massless. What does this imply about the tension forces on the right and left side of the pulley?
- Using energy ideas, what is the translational speed of the masses m_1 and m_2 after m_1 has fallen by an amount $|\Delta y| = y = 1.0m$, from rest?
- What is the rotational speed ω of the pulley after after m_1 has fallen by an amount $|\Delta y| = y = 1.0m$, from rest?
- Through how much angle $\Delta\theta$ (in radians) has the pulley rotated after m_1 has fallen by an amount $|\Delta y| = y = 1.0m$, from rest?

Rotational Motion

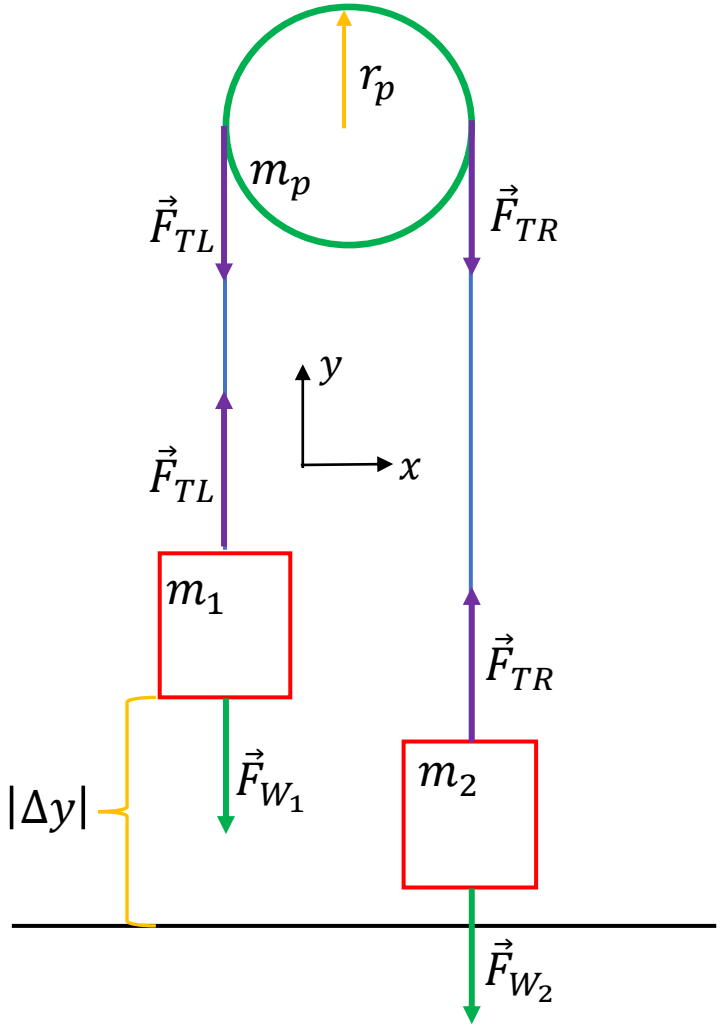
Example 4 Solution:

When we started forces a while ago, we said that if the pulley has no mass, then the tension forces on either side of the pulley were the same, without any justification or proof.

Of course, in order for the pulley to spin in some particular direction the tension forces cannot be equal to each other in magnitude. Depending on which way the pulley spins, one of the tension forces in the ropes must be greater than the other.

We're now able to prove the assumption that if the pulley has no mass, then the tension forces are equal. Of course, for the pulley to spin they cannot be equal and of course, there's no such thing as a massless pulley.

Since the pulley is massive, let's call the tension force in the rope on the left side of the pulley $\vec{F}_{TL}$ and the tension force in the right side of the pulley $\vec{F}_{TR}$.



Rotational Motion

Example 4 Solution continued:

- We have to start somewhere, so let's start with m_1.
- The forces that act on m_1 in the vertical direction from Newton's laws of motion:

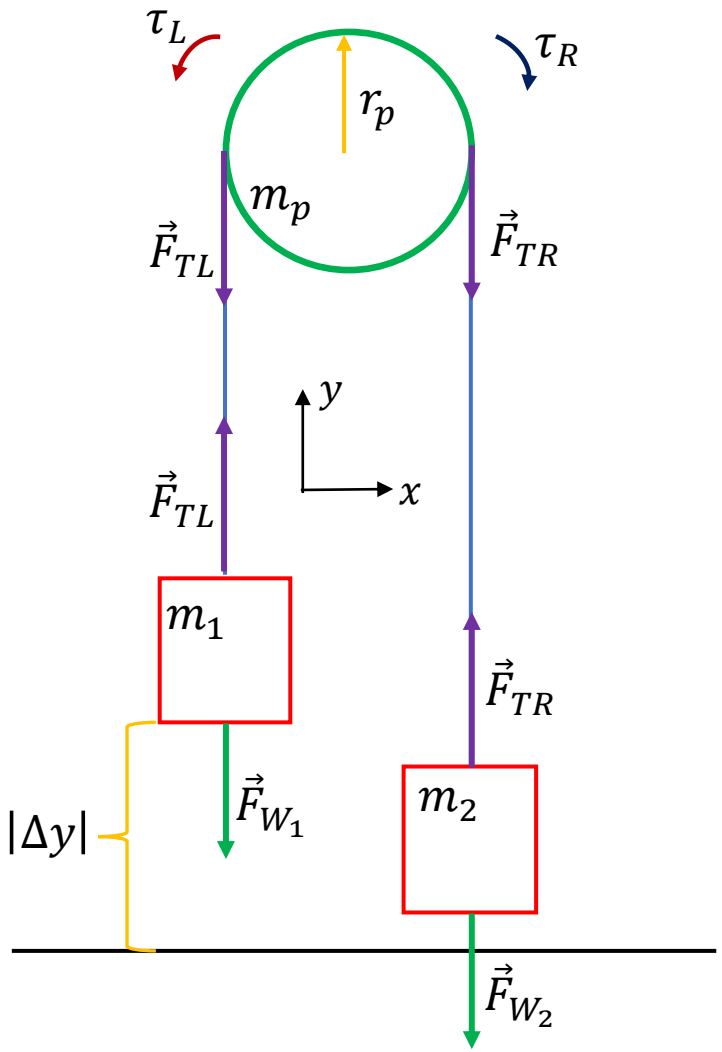
$$F_{TL} - F_{W1} = F_{TL} - m_1 g = -m_1 a \rightarrow F_{TL} = m_1 g - m_1 a$$

- The forces that act on m_2 in the vertical direction from Newton's laws of motion:

$$F_{TR} - F_{W2} = F_{TR} - m_2 g = m_2 a \rightarrow F_{TR} = m_2 g + m_2 a$$

- These two expressions both contain an unknown acceleration and two unknown tension forces. To determine the tension forces let's look at the torques about the axis of rotation of the pulley. Again, we take that counterclockwise rotations are positive.

$$\tau_L + \tau_R = r_p F_{TL} \sin 90 - r_p F_{TR} \sin 90 = r_p F_{TL} - r_p F_{TR} = I_p \alpha_p = \left(\frac{1}{2} m_p r_p^2\right) \alpha$$



Rotational Motion

Example 4 Solution continued:

- The tangential acceleration of a piece of string off the edge of the pulley (a) is related to the angular acceleration of the pulley about the axis of rotation (α) and where we want to calculate the acceleration (at the outer edge of the pulley at r_p) by $a = r_p\alpha$.

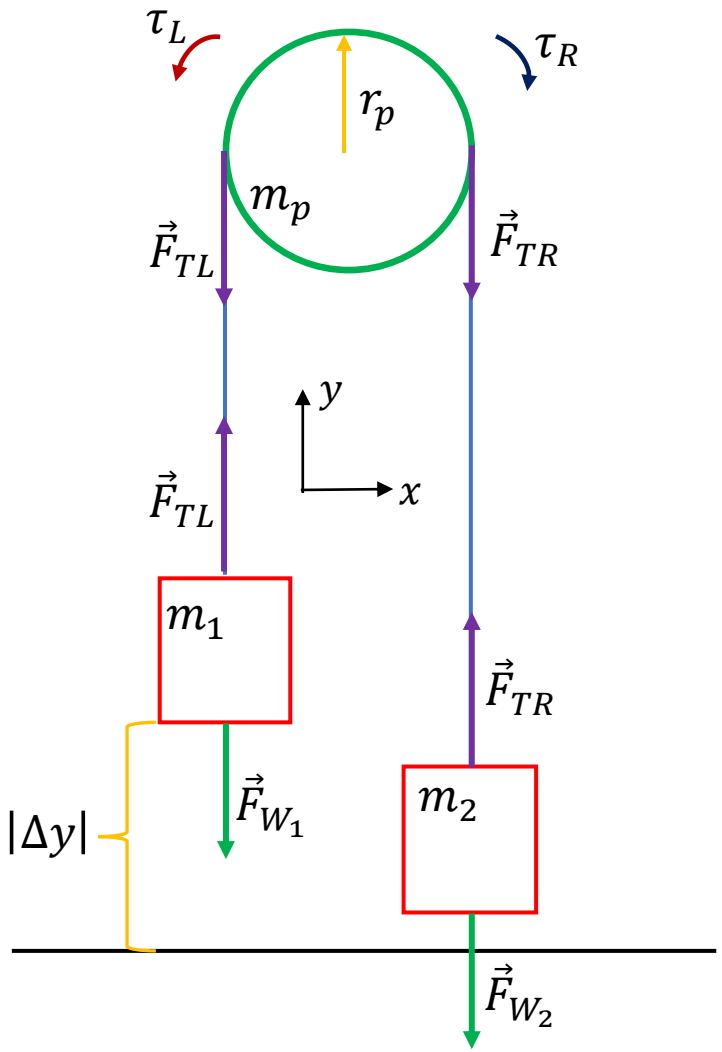
- Substituting this into our equation for the torques we can solve for the difference between the left and right tension forces.

$$r_p F_{TL} - r_p F_{TR} = I_p \alpha_p = \left(\frac{1}{2} m_p r_p^2\right)\alpha = \left(\frac{1}{2} m_p r_p^2\right)\frac{a}{r_p}$$

$$\rightarrow \boxed{F_{TL} - F_{TR} = \frac{1}{2} m_p a}$$

- Here the difference in the tension forces on either side of the pulley cause the pulley to rotate. Here the pulley rotates counterclockwise (as m_1 falls) thus $F_{TL} > F_{TR}$.

- In addition, if the pulley is massless ($m_p = 0$) we see that $F_{TL} = F_{TR} = F_T$, which was our original assumption back in chapter 4.



Rotational Motion

Example 4 Solution continued:

- Now we can solve for the acceleration of the system.

$$F_{TL} - F_{TR} = \frac{1}{2}m_p a$$

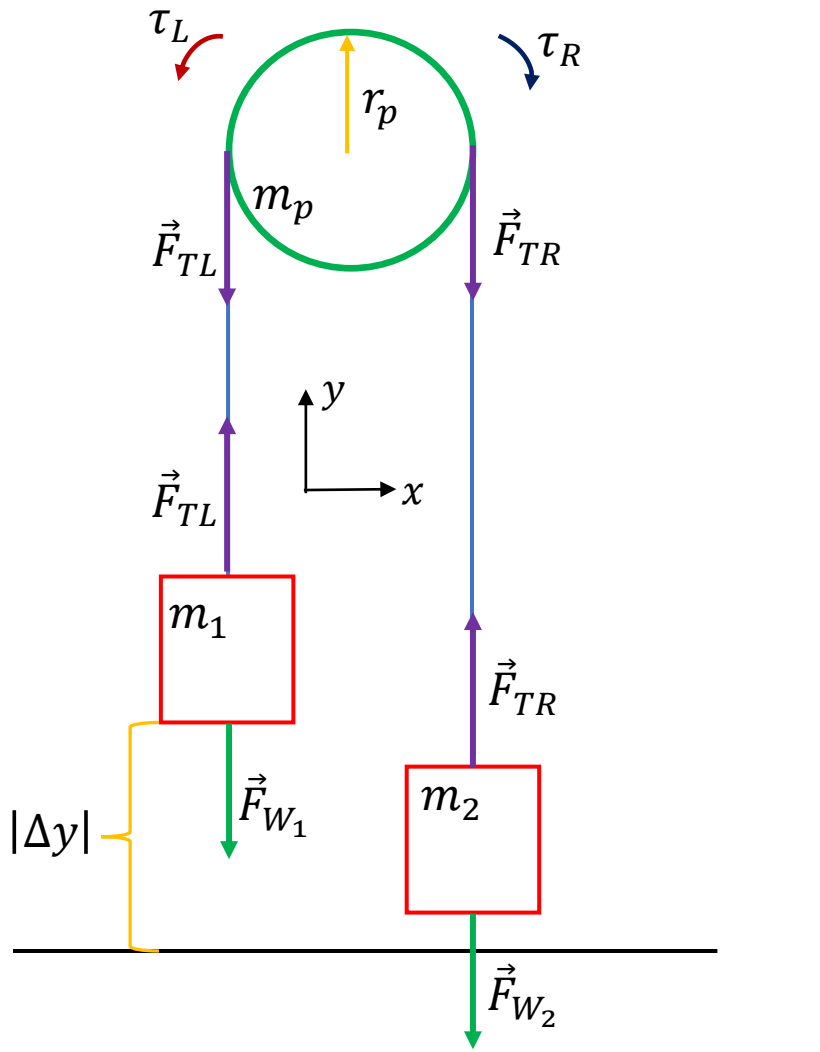
$$(m_1 g - m_1 a) - (m_2 g + m_2 a) = \frac{1}{2}m_p a$$

$$\rightarrow \boxed{a = \left(\frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}m_p}\right) g}$$

- If the pulley was massless ($m_p = 0$), the acceleration is the same as that we calculated previously when we assumed that the tension forces were the same in the rope.

- Since forces and torques are constant we can use our time-independent equation of motion to determine the speed of the system of masses.

- We'll come back again and re-solve this problem using conservation of energy.



$$v_{fy}^2 = v_{iy}^2 + 2a_y\Delta y \rightarrow \boxed{v_{fy} = \sqrt{2\left(\frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}m_p}\right) g\Delta y}}$$

Rotational Motion

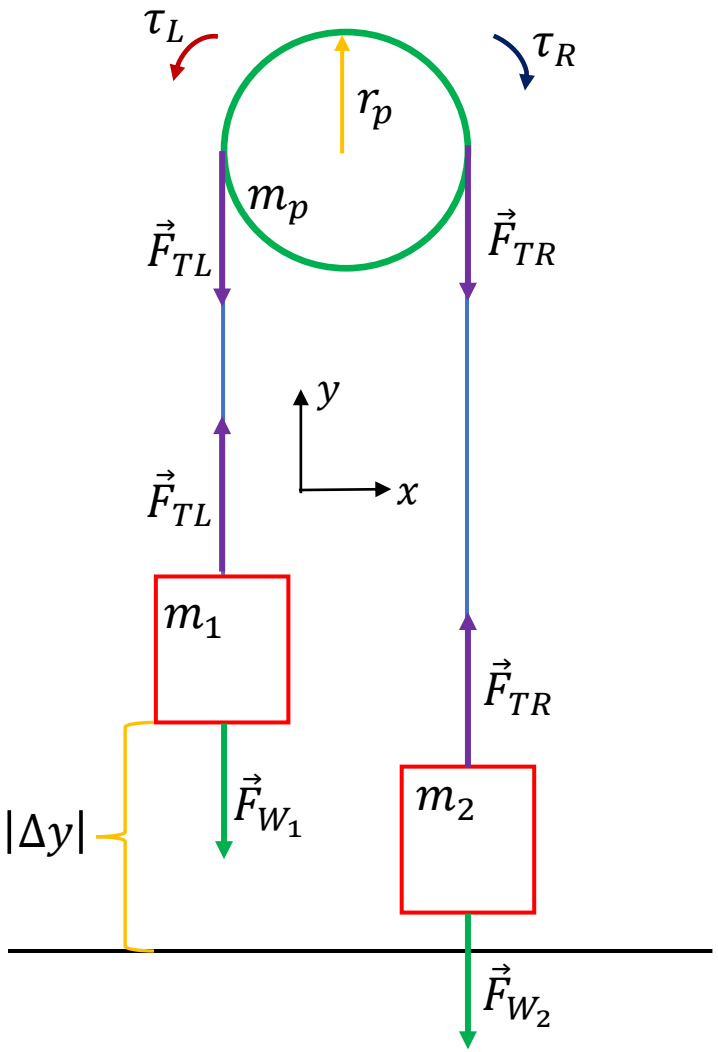
Example 4 Solution continued:

The acceleration of the system:

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}m_p}\right) g = \left(\frac{2kg - 1kg}{2kg + 1kg + 0.5 \cdot 0.25kg}\right) \cdot 9.8\frac{m}{s^2} = 3.14\frac{m}{s^2}$$

The speed of the of the system of masses:

$$v_{fy} = \sqrt{2\left(\frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}m_p}\right) g\Delta y} = \sqrt{2 \cdot 3.14\frac{m}{s^2} \cdot 1m} = 2.5\frac{m}{s}$$



Rotational Motion

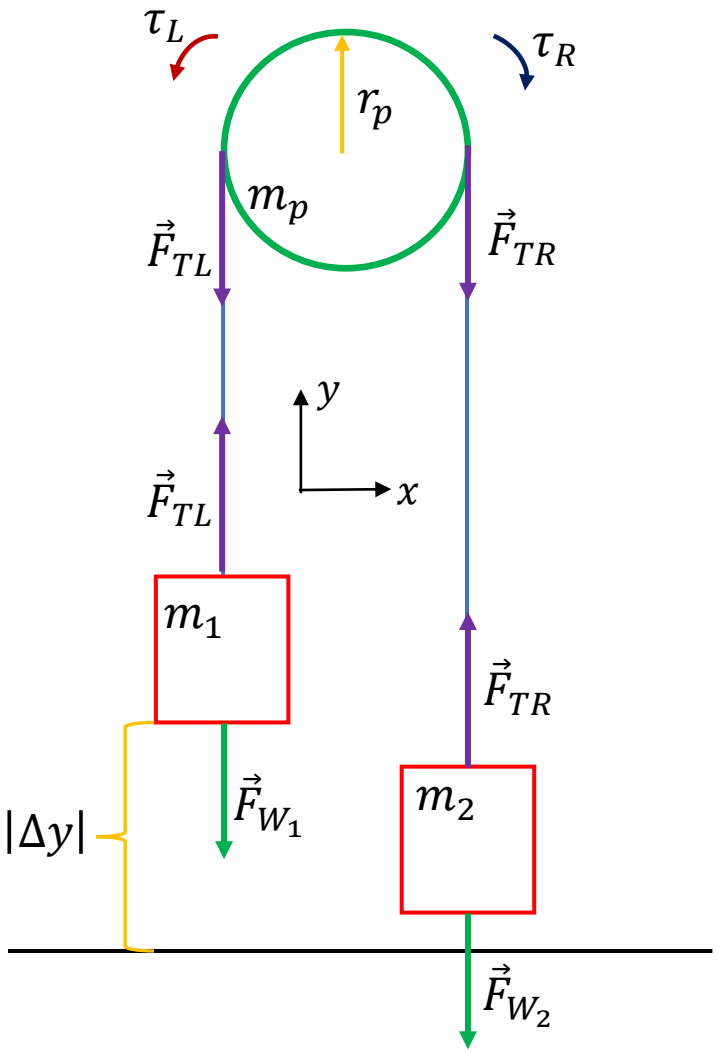
Example 4 Solution continued:

The acceleration of the system:

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}m_p}\right) g = \left(\frac{2kg - 1kg}{2kg + 1kg + 0.5 \cdot 0.25kg}\right) \cdot 9.8\frac{m}{s^2} = 3.14\frac{m}{s^2}$$

The speed of the of the system of masses:

$$v_{fy} = \sqrt{2\left(\frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}m_p}\right) g\Delta y} = \sqrt{2 \cdot 3.14\frac{m}{s^2} \cdot 1m} = 2.5\frac{m}{s}$$



Rotational Motion

Example 4 Solution continued:

By conservation of energy, we can calculate the speed of either m_1 falling or m_2 rising.

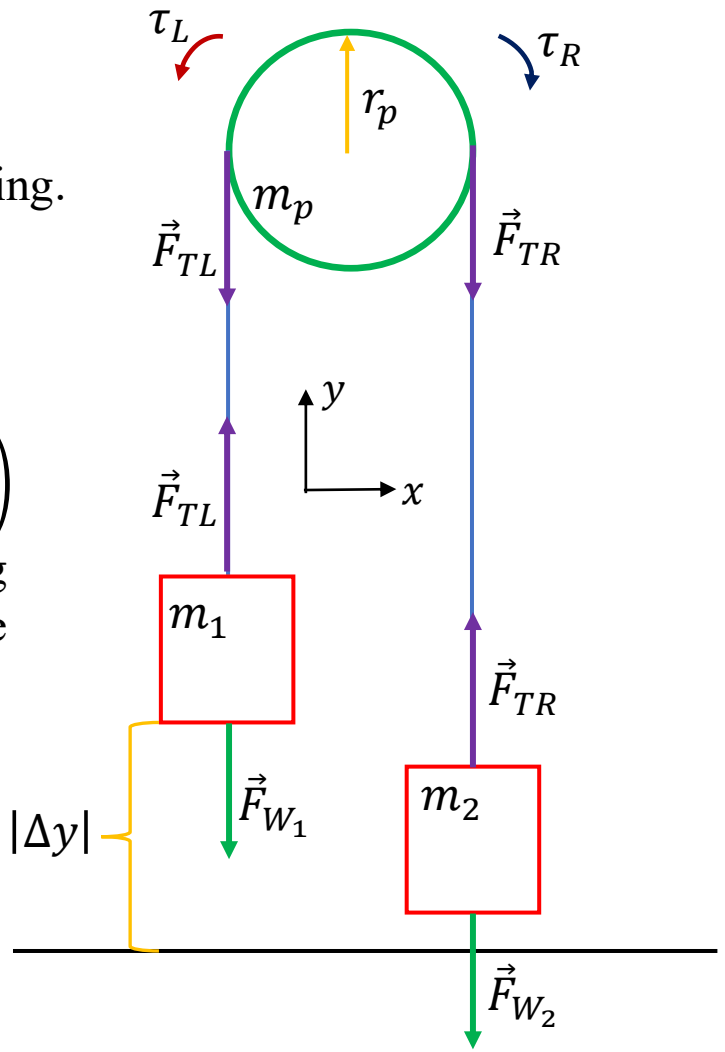
$$\Delta E_{system} = \cancel{W_{fr}}^{0} = \Delta K_1 + \Delta U_{g1} + \Delta K_2 + \Delta U_{g2} + \Delta K_R + \Delta U_S$$

$$0 = \left(\frac{1}{2}m_1 v_f^2 - 0\right) + (0 - m_1 g y) + \left(\frac{1}{2}m_2 v_f^2 - 0\right) + (m_2 g y - 0) + \left(\frac{1}{2}I\omega_f^2 - 0\right)$$

To determine the translational speed, we relate the translational speed of the string coming off the left side of the pulley (or onto the right side of the pulley) to the rotational speed of the pulley by $v_f = r_p\omega_f$.

$$(m_1 - m_2)gy = \frac{1}{2}\left(m_1 + m_2 + \left(\frac{\frac{1}{2}m_p r_p^2}{r_p}\right)\right)v_f^2$$

$$v_{fy} = \sqrt{2\left(\frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}m_p}\right)g\Delta y} = \sqrt{2 \cdot 3.14\frac{m}{s^2} \cdot 1m} = 2.5\frac{m}{s}$$

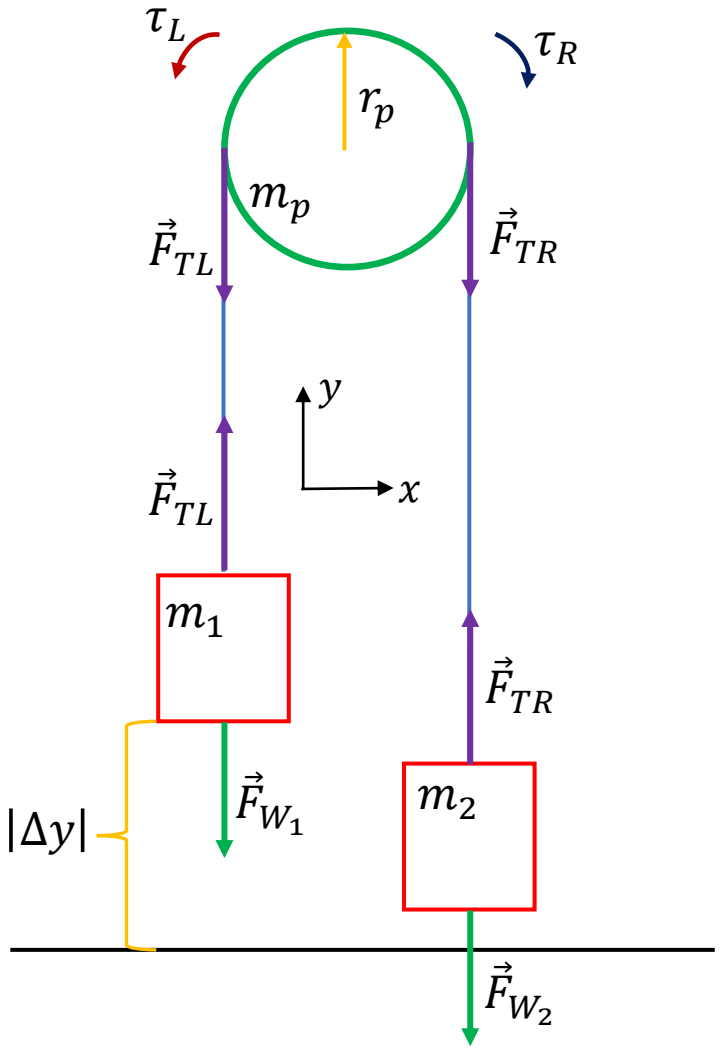


Rotational Dynamics

Example 5:

Suppose a box of mass $m_1 = 2kg$ is released from rest and allowed to fall through a height of $|\Delta y| = 1.0m$. Assume m_1 is connected to a mass $m_2 = 1kg$ by a massless rope that passes over a massive pulley with mass $m_p = 0.25kg$ and radius $r_p = 0.1m$.

- Using the acceleration calculated in the last problem, how long does it take for mass m_1 to fall through $|\Delta y| = 1.0m$ starting from rest?
- What are the translational speed of the system of masses m_1 and m_2 and the rotational speed of the pulley after this time? Do this two ways.
- How many revolutions has the pulley spun through when mass m_1 has fallen by $|\Delta y| = 1.0m$? Use the time calculated above and do it two ways.



Rotational Motion

Example 5 Solution:

- Since the net force (and torque) are constant let's use our equation of motion for constant acceleration to determine the motion parameters.

- The translational acceleration of the system of masses is:

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}m_p}\right)g = \left(\frac{2kg - 1kg}{2kg + 1kg + \frac{1}{2}(0.25kg)}\right)\times 9.8\frac{m}{s^2} = 3.14\frac{m}{s^2}$$

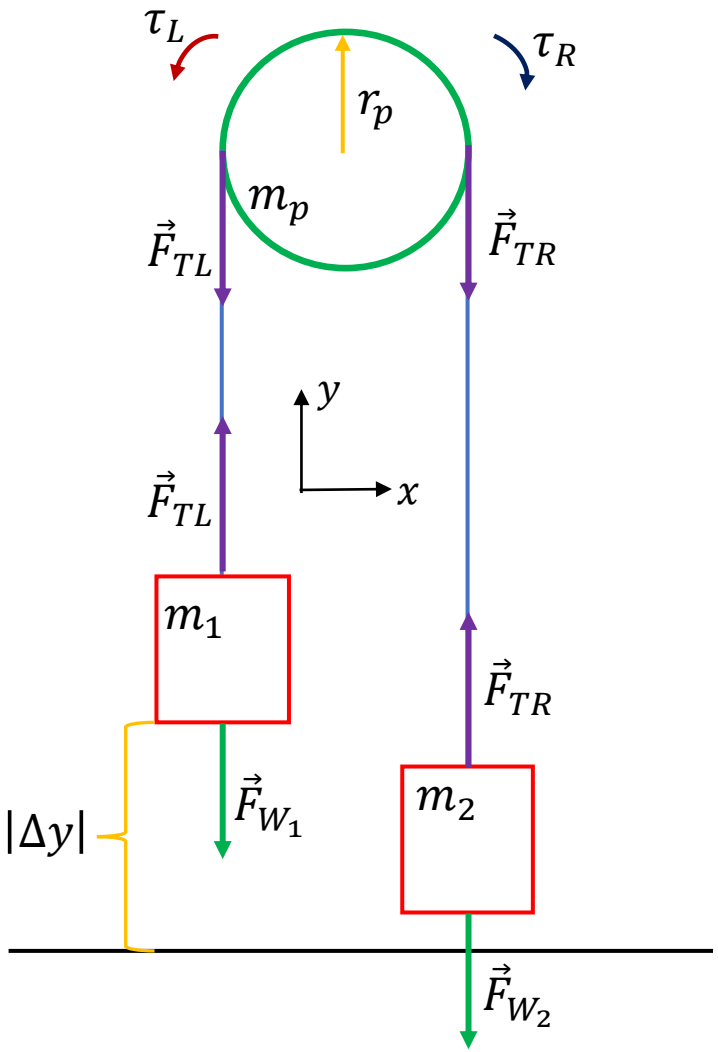
- The time it takes m_1 to fall through a height of $|\Delta y| = 1.0m$ starting from rest can be determined from the vertical trajectory equation of motion.

$$y_f = \overset{0}{y_i} + \overset{0}{v_{iy}}t + \frac{1}{2}a_y t^2 \rightarrow t = \sqrt{\frac{2y_i}{a_y}} = \sqrt{\frac{2\times 1m}{3.14\frac{m}{s^2}}} = 0.8s$$

- The translational speeds of masses and can be found in two ways. One using the time-dependent equation of motion and two, using the time-independent equation of motion.

$$v_{fy} = v_{iy} + a_y t \rightarrow |v_{fy}| = |a_y|t = 3.14\frac{m}{s^2}\times 0.8s = 2.51\frac{m}{s}$$

$$v_{fy}^2 = v_{iy}^2 + 2a_y\Delta y \rightarrow v_{fy} = \sqrt{2a_y\Delta y} = \sqrt{2\times 3.14\frac{m}{s^2}\times 1m} = 2.51\frac{m}{s}$$



Rotational Motion

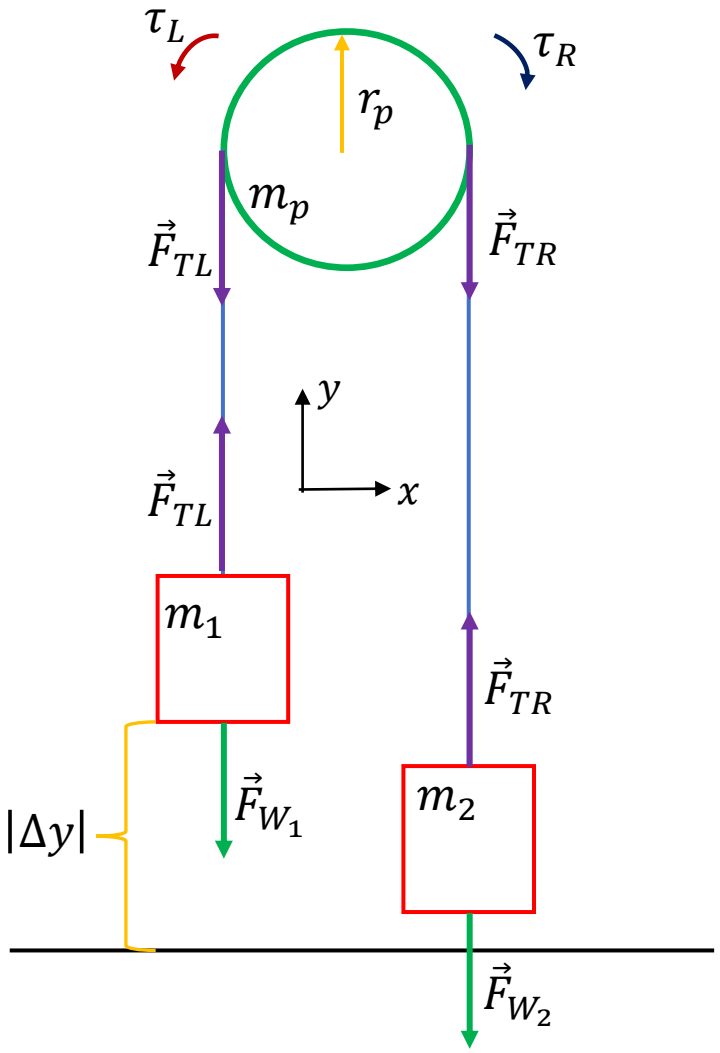
Example 5 Solution continued:

- The rotational speed of the pulley can be determined in three ways.
- The first method is to use the relation between the tangential speed of the rope off of the edge of the pulley (which equals the translational speed of the masses) and the radius of the pulley, r_p.

$$v = r_p\omega \rightarrow \omega = \frac{v}{r_p} = \frac{2.51\frac{m}{s}}{0.1m} = 25.1\frac{rad}{s}$$

- The second and third methods rely on the angular acceleration of the pulley (α) and the angular displacement of the pulley, $\Delta\theta$.
- The angular acceleration of the pulley is related to the translational acceleration of a piece of rope off of the edge of the pulley (at r_p).

$$a = r_p\alpha \rightarrow \alpha = \frac{a}{r_p} = \frac{3.14\frac{rad}{s}}{0.1m} = 31.4\frac{rad}{s^2}$$



Rotational Motion

Example 5 Solution continued:

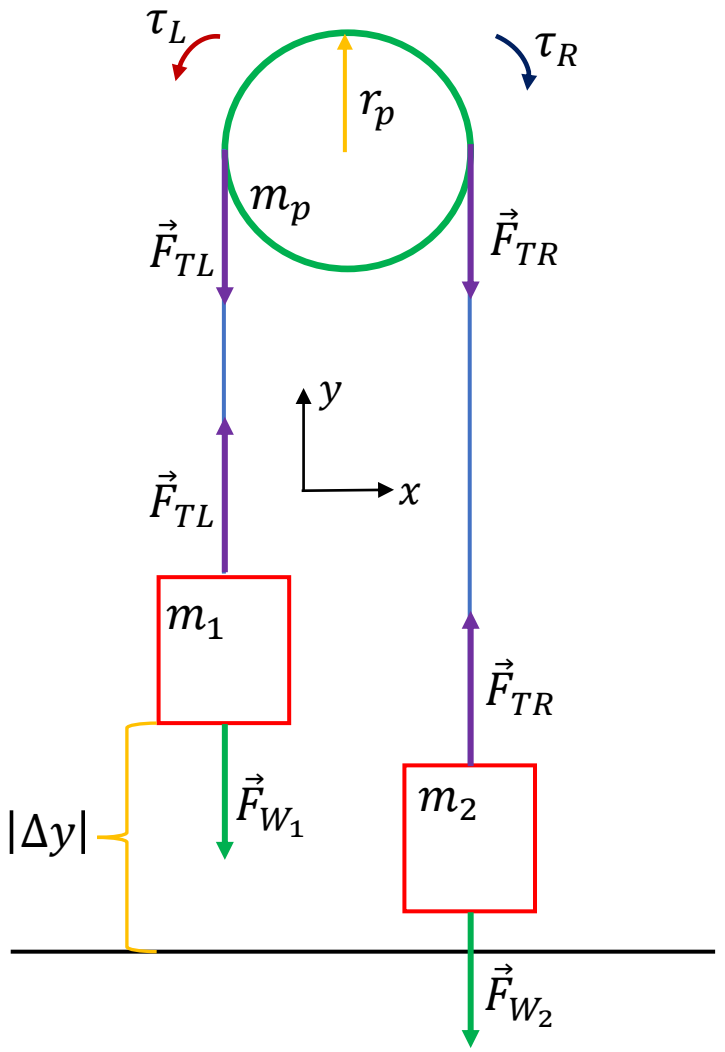
- The second method to determine the rotational speed of the pulley is by using the time-dependent rotational velocity equation of motion.

$$\omega_f = \cancel{\omega_i}^{0} + \alpha t \rightarrow \omega_f = \alpha t = 31.4\frac{rad}{s^2}\times 0.8s = 25.1\frac{rad}{s}$$

- The third method is by using the time-independent velocity equation of motion.

$$\omega_f^2 = \cancel{\omega_i^2}^{0} + 2\alpha\Delta\theta \rightarrow \omega_f = \sqrt{2\alpha\Delta\theta}$$

- Here we need to know how much angle $\Delta\theta$ the pulley has rotated through.
- The angular displacement $\Delta\theta$ of the pulley about its axis of rotation can be determined in two ways.
- One is using the relation between the amount of rope that comes off of the edge of the pulley and the radius of the pulley.



Rotational Motion

Example 5 Solution continued:

- Relating the amount of rope that comes off the pulley to the rotational displacement of the pulley we have:

$$s = y = r_p\theta \rightarrow \theta = \frac{y}{r_p} = \frac{1.0m}{0.1m} = 10rad$$

- This equates to some number of revolutions of the pulley. To determine the number of revolutions, we note that every $2\pi\ rad$ the pulley makes one revolution.

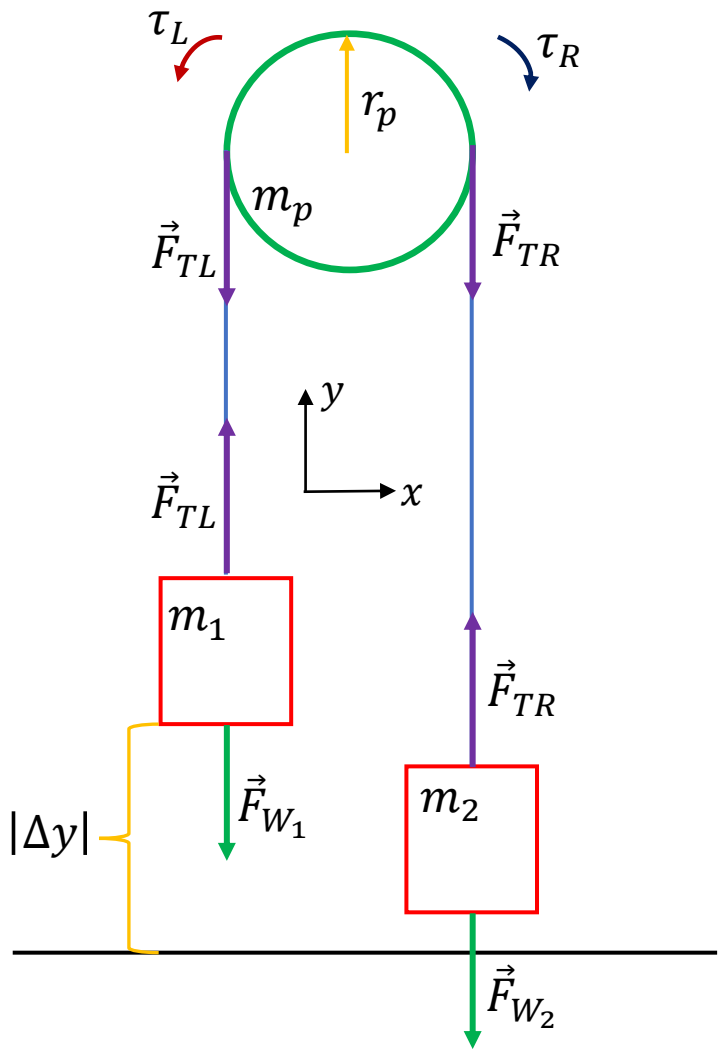
$$\#\text{revolutions} = 10rad \times \frac{1rev}{2\pi rad} = 1.59\text{revolutions}$$

- We could also have calculated the angular displacement of the pulley using our angular trajectory equation of motion.

$$\theta_f = \theta_i + \cancel{\omega_i^{0}}t + \frac{1}{2}\alpha t^2 \rightarrow \Delta\theta = \frac{1}{2}\alpha t^2 = \frac{1}{2}\left(31.4\frac{rad}{s^2}\right)(0.8s)^2 = 10rad$$

- And lastly, the third method to determine the rotational speed of the pulley was by using the time-independent velocity equation of motion.

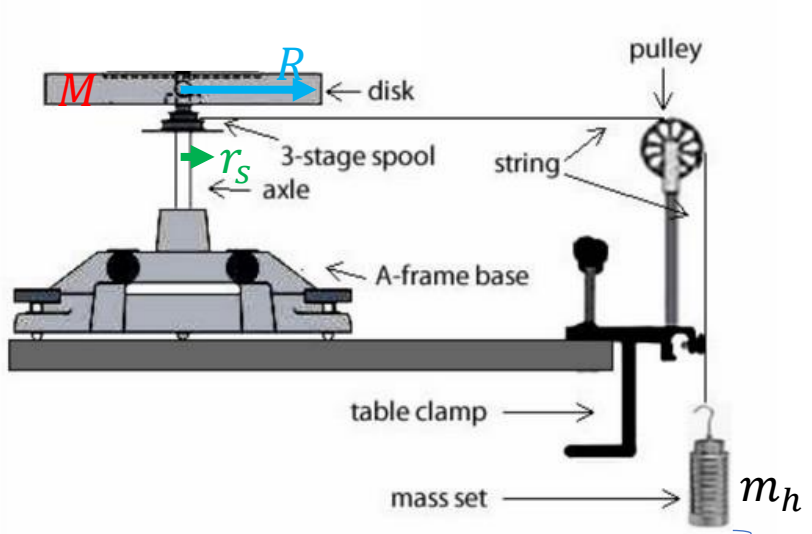
$$\omega_f^2 = \cancel{\omega_i^{0}}^2 + 2\alpha\Delta\theta \rightarrow \omega_f = \sqrt{2\alpha\Delta\theta} = \sqrt{2\times 31.4\frac{rad}{s^2}\times 10rad} = 25.1\frac{rad}{s}$$



Rotational Dynamics

Example 6:

The expressions that we've been using for the moment of inertia of various objects in solving problems we've taken from Table 7.1 on page 226 of KJF College Physics: A Strategic Approach, Volume 1. These moments of inertia are experimentally determined. To experimentally determine the moment of inertia of an object, we use this apparatus on the right. Here, we'll determine an expression for the moment of inertia of a disk. Consider a disk of radius R and mass M mounted on a frictionless axle. A light string is wound around the three-stage spool and let the spool around which the string is wound has a radius r_s. A set of masses are used to create different torques about the spool. Each separate mass (m_h) is attached to the free end of the string and is released from rest. Each hanging mass on the end of the sting falls through a height y. The pulley in the system is considered massless (or at least so much less in mass than any other mass in the system).



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- Using the forces and torques that act in the system, what is the expression for the moment of inertia of the disk?

- Using energy ideas, determine an expression for the moment of inertia of the disk.

Rotational Dynamics
- Conservation of Energy

Example 6 Solution :

- Let's first solve this problem by using torques and forces.

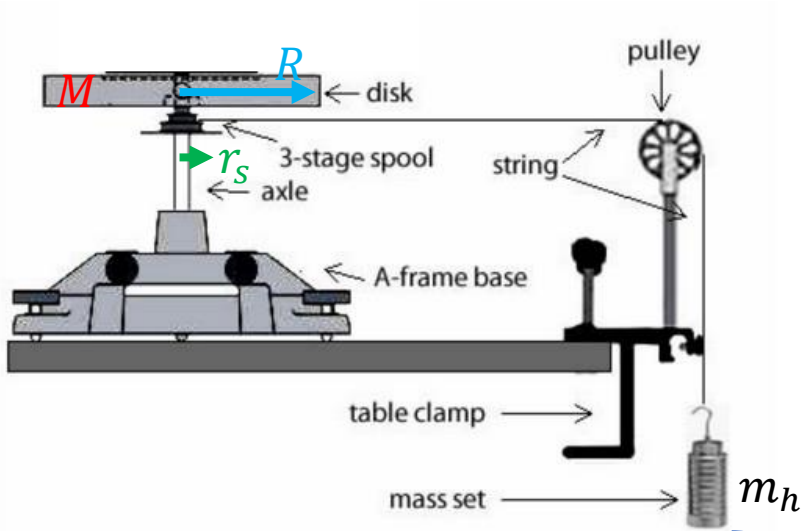
- We'll redraw the system (below right) so that we can see how the forces act in the system. This is a combination of an overhead view (of the spool/axle) and a face on view of the hanging mass.

- The tension force that acts on the spool creates a torque on the system and causes it to spin. This in turn spins the disk connected to the axle and the system accelerates about the axis of rotation (which is parallel to the center of the axle).

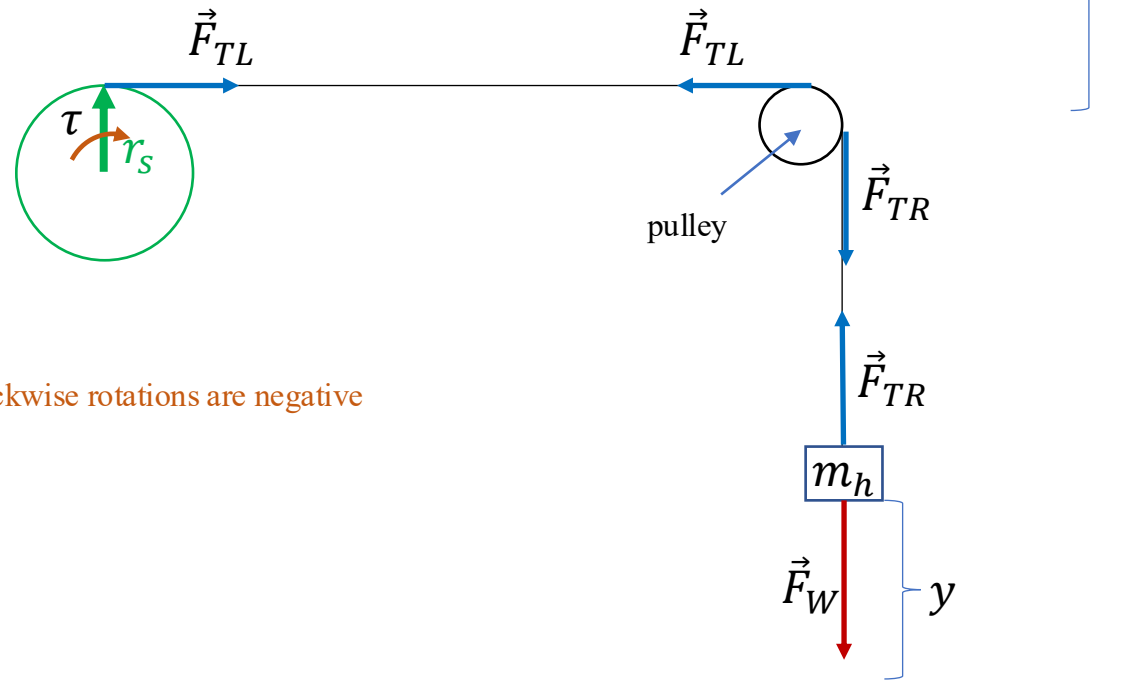
$$\tau = -r_s F_{TL} \sin\theta = -r_s F_{TL} \sin 90 = -r_s F_{TL} = -I_D \alpha$$

Clockwise rotations are negative

- In order to calculate the moment of inertia of the system (I_D) we need to know the expression for the tension force in the string F_{TL} and the angular acceleration of the system, α.



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Rotational Dynamics
- Conservation of Energy

Example 6 Solution continued:

- To determine the tension force that acts on the spool, let's look at the pulley.)

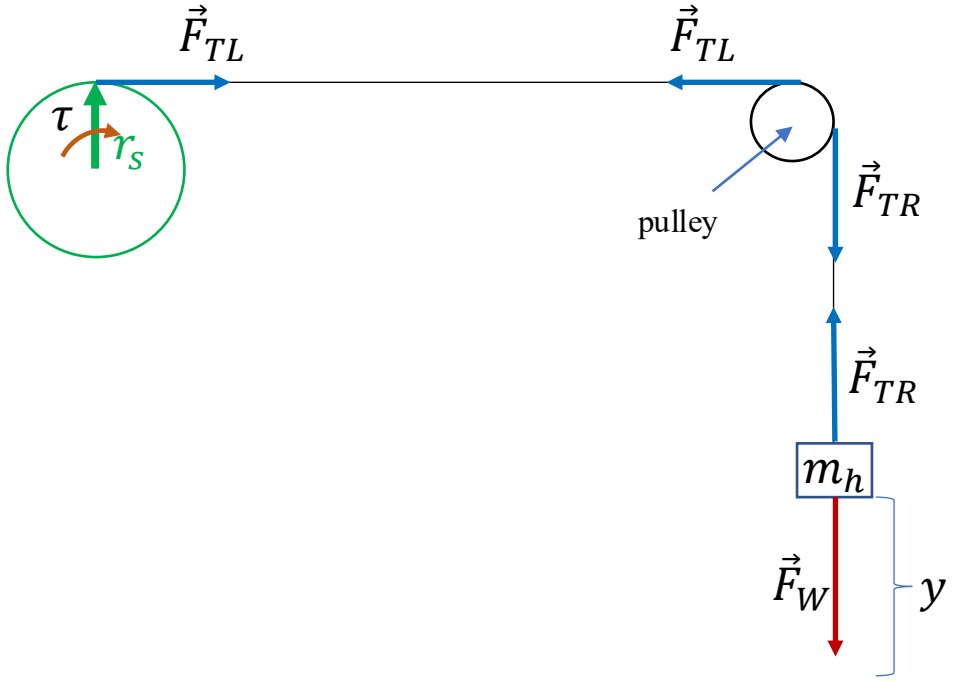
- The tension forces on either side of the pulley cause the pulley to rotate (clockwise) about the axis of rotation of the pulley. Assuming that the pulley has a radius r_p and taking counterclockwise as the positive direction for the torque we have

$$\tau_L - \tau_R = r_p F_{TL} - r_p F_{TR} = I_p \alpha = \left(\frac{1}{2} m_p r_p^2\right) \alpha$$

- We assume that the pulley's mass is very small so that $m_p \sim 0$.

$$\tau_L - \tau_R = r_p F_{TL} - r_p F_{TR} = I_p \alpha = \left(\frac{1}{2} \overset{0}{m_p} r_p^2\right) \alpha = 0 \rightarrow \tau_L = \tau_R \rightarrow F_{TL} = F_{TR}$$

- To determine the tension in the right side of the pulley, F_{TR} (which is the tension force in the string) we look at the forces that act on the hanging mass.



Rotational Dynamics
- Conservation of Energy

Example 6 Solution continued:

- The tension in the right side of the pulley, F_{TR} assuming a standard cartesian coordinate system:

$$F_{TR} - F_W = F_{TR} - m_h g = -m_h a$$

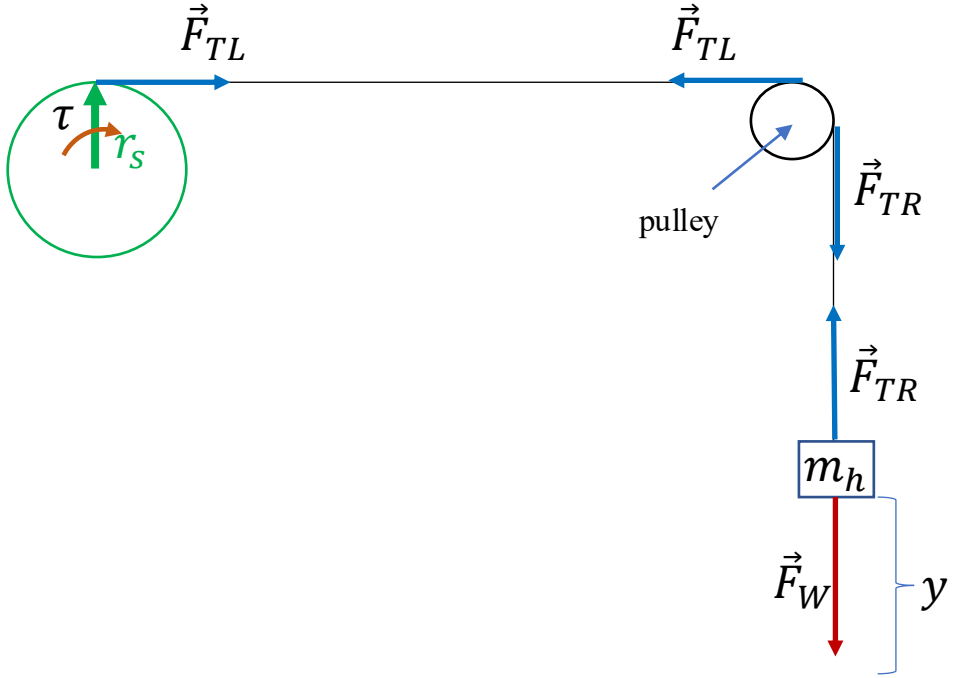
$$\longrightarrow \quad F_{TR} = m_h g - m_h a$$

- Since $F_{TR} = F_{TL}$ we have in our equation for the torque about the axle:

$$\tau = -r_s F_{TL} = -r_s(m_h g - m_h a) = -I_D \alpha$$

- The tangential acceleration of the piece of string off of the edge of the spool is related to the angular acceleration of the axle about its axis of rotation by $a = r_s \alpha$.

$$r_s(m_h g - m_h a) = I_D \alpha = I_D \left(\frac{a}{r_s}\right) \rightarrow \boxed{I_D = m_h r_s^2 \left(\frac{g}{a} - 1\right)}$$



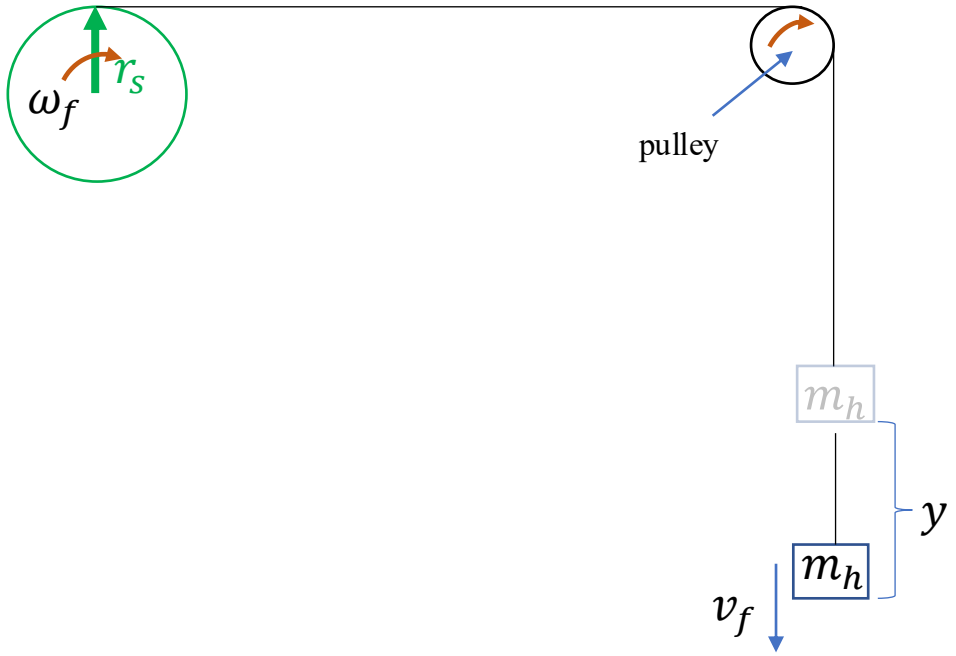
For a given m_h, a measurement of the acceleration a of the hanging mass m_h will let us determine the moment of inertia of the disk, I_D.

Notice also that all moments of inertia look like the product of a constant and mr^2, which is the form we have here.

Rotational Dynamics
- Conservation of Energy

Example 6 Solution continued:

- Now let's solve this problem by using energy ideas.
- As the hanging mass falls the string connecting the hanging mass and the axle pulls on the axle and the axle with the disk begins to spin.
- We'll apply conservation of energy (assuming there's no work done by any frictional torques in the axle).
- The system starts from rest translationally ($v_i = 0$) and rotationally ($\omega_i = 0$).
- The disk does not translate through space or change height, it only rotates. Thus, $\Delta K_{TD} = 0$, $\Delta U_{gD} = 0$, and $\Delta K_{RD} \neq 0$.
- The hanging mass falls and translates. For the hanging mass we have $\Delta K_{Tm_h} \neq 0$ and $\Delta U_{gm_h} \neq 0$.



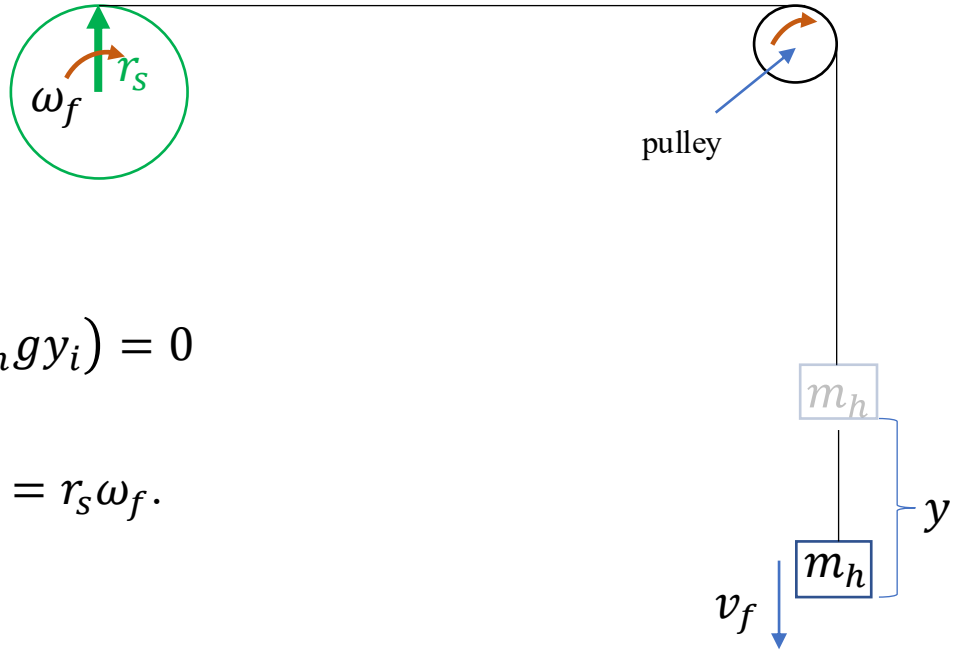
Rotational Dynamics
- Conservation of Energy

Example 6 Solution continued:

$$\Delta E = \Delta K_{RD} + \Delta K_{Tm_h} + \Delta U_{gm_h} + \Delta U_s = 0$$

$$\Delta E = \left(\frac{1}{2}I_D\omega_f^2 - \frac{1}{2}I_D\omega_i^2\right) + \left(\frac{1}{2}m_h v_f^2 - \frac{1}{2}m_h v_i^2\right) + \left(m_h g y_f - m_h g y_i\right) = 0$$

- The translational speed v_f of the hanging mass as it falls is related to the rotational speed ω_f of string off of the spool at the spools radius r_s, by $v_f = r_s\omega_f$.



- Our statement of conservation of energy is therefore:

$$\Delta E = \frac{1}{2}I_D\omega_f^2 + \frac{1}{2}m_h v_f^2 - m_h g y_i = \frac{1}{2}I_D\left(\frac{v_f}{r_s}\right)^2 + \frac{1}{2}m_h v_f^2 - m_h g y = 0$$

- Solving for the moment of inertia of the disk we find:

$$\rightarrow \quad \boxed{I_D = m_h r_s^2\left(\frac{v_f^2}{2gy} - 1\right)}$$

For a given m_h, a measurement of the translational speed v_f of the hanging mass m_h will let us determine the moment of inertia of the disk, I_D.

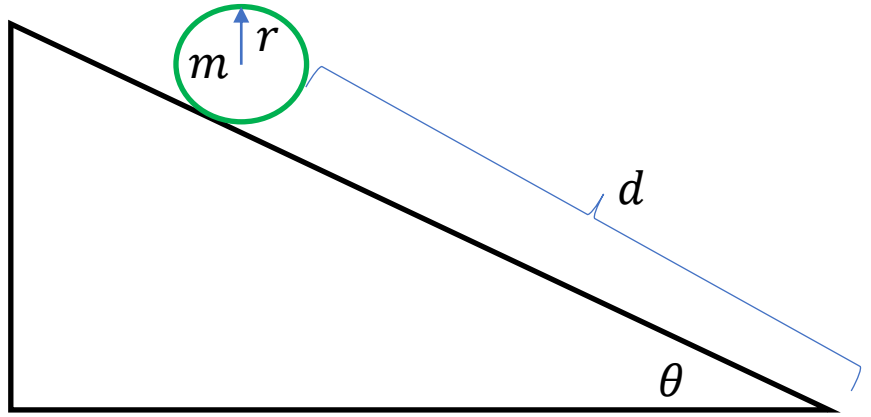
Notice also that all moments of inertia look like the product of a constant and mr^2, which is the form we have here.

Rotational Dynamics

Example 7:

We're in a position now to talk about why objects roll. Consider several objects (a hoop, a solid cylinder, and a solid ball) each of mass m and equal radii $r_{hoop} = r_{cylinder} = r_{ball} = r$. All the objects are placed on an incline (oriented at an angle of inclination θ measured with respect to the horizontal) and released from rest both translationally and rotationally. Static friction between the object and the ramp is responsible for the rolling motion of the object, due to a torque created about the center-of-mass of the object. If it were not for friction, there would be no motion except for sliding down the incline by the object. We need static friction (which acts at a point) to cause the object to turn on its axis.

- What is the expression for the translational acceleration of each of the objects?
- What is the speed of each of the objects at the bottom of the incline if each object starts from rest and covers a distance d along the incline. Do this using forces/torques and then using energy.
- What is the order of arrival of the objects at the bottom of the incline? Which gets there first, second and third? Or do they all get there at the same time?



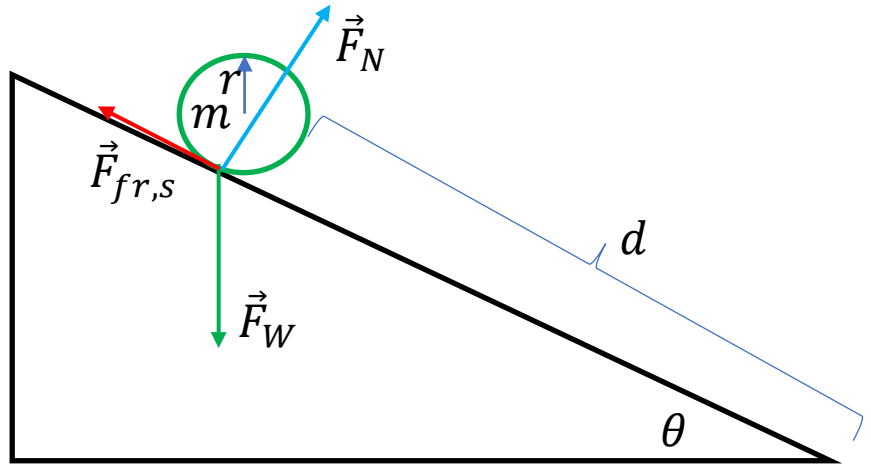
Rotational Motion

Example 7 Solution:

- In the absence of friction, the objects slide down the ramp. They would not roll. Friction creates a torque in the system, and the net external torque causes the objects to rotate.
- Let's put the forces that act on the system on our diagram.
- Assuming a tilted coordinate system and applying Newton's laws of motion parallel to the incline we get:

$$-F_{fr,s} + F_{Wx} = -F_{fr,s} + mg\sin\theta = ma_x$$

- Now, each object is momentarily at rest at each point along the incline with respect to the incline where the object touches the incline.
- The force that causes the torque is static friction.
- Now, I don't know if the force of static friction here can be written as the product of the coefficient of friction and the normal force or not. Since we're dealing with rolling motion, maybe it can be, maybe it can't. Let's just leave it as $\vec{F}_{fr,s}$. In the end, it turns out that it won't matter.



Rotational Motion

Example 7 Solution continued:

- The torque created by the static frictional force is clockwise in direction (by the right-hand rule, taken as negative) and is given by:

$$\tau = rF\sin\phi = -rF_{fr,s}\sin 90 = -rF_{fr,s}$$

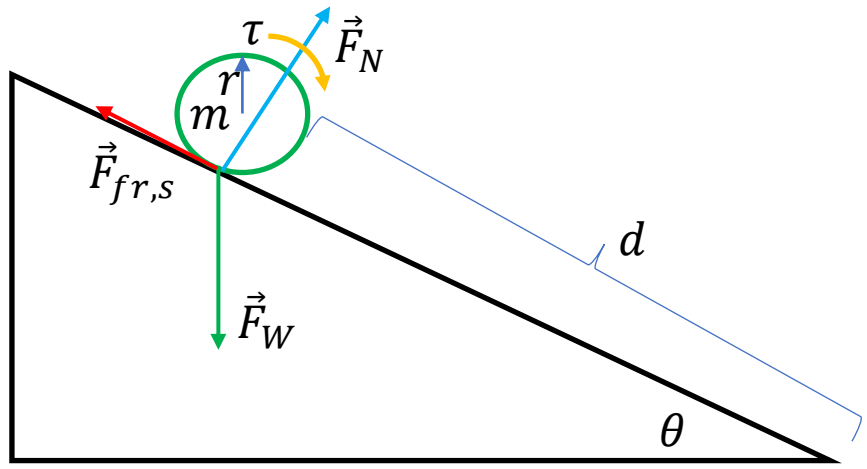
- The torque, by Newton's laws of motion is related to the moment of inertia (I) and the angular acceleration of the object about the axis of rotation (α).

- Further, the angular acceleration of the object about the axis of rotation (α) is related to the acceleration of the system down the incline by $a = r\alpha \rightarrow \alpha = \frac{a}{r}$.

$$\tau = I\alpha \rightarrow -rF_{fr,s} = -I\alpha = -I\frac{a}{r}$$

- We can replace the static frictional force in terms of the acceleration of the object down the incline and solve for the acceleration of the object down the incline.

$$-F_{fr,s} + mg\sin\theta = -I\frac{a}{r^2} + mg\sin\theta = ma \longrightarrow \boxed{a = \frac{g\sin\theta}{1 + \frac{I}{mr^2}}}$$



Rotational Motion

Example 7 Solution continued:

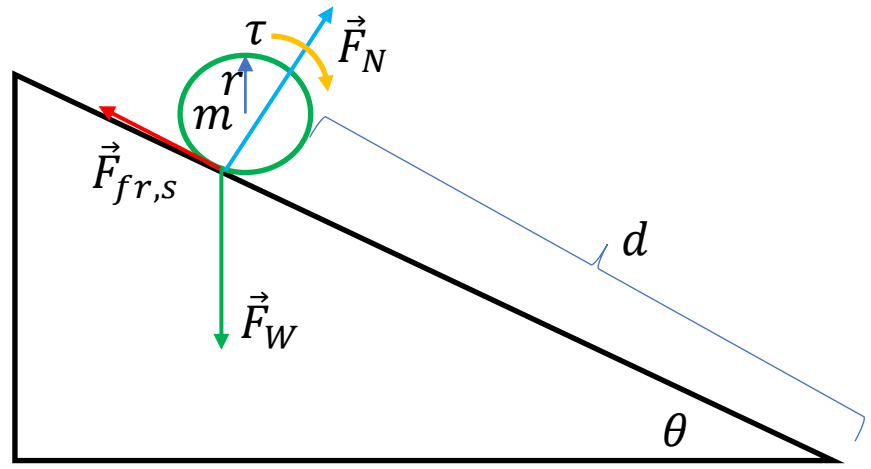
- We can write expressions for the acceleration of each object down the incline, where we look up the moment of inertia from figure 8.20 for the hoop (rotated about an axis through its center), cylinder (rotated about an axis through its center), and ball (rotated about any axis through its center).

$$a_{hoop} = \frac{g\sin\theta}{1+\frac{mr^2}{mr^2}} = \frac{1}{2}g\sin\theta \qquad a_{cylinder} = \frac{g\sin\theta}{1+\frac{\frac{1}{2}mr^2}{mr^2}} = \frac{2}{3}g\sin\theta \qquad a_{ball} = \frac{g\sin\theta}{1+\frac{\frac{2}{5}mr^2}{mr^2}} = \frac{5}{7}g\sin\theta$$

- Since the acceleration is constant (due to the forces and torques being constant) we can calculate the speed of each object at the bottom of the incline. Let's use the time-independent equation of motion.

$$v_{fx}^2 = v_{ix}^2 + 2a_x\Delta x \rightarrow v_{fx} = \sqrt{2ad}$$

$$\rightarrow \begin{cases} v_{fx,hoop} = \sqrt{2\times\frac{1}{2}g\sin\theta\times d} = \sqrt{gd\sin\theta} \\ v_{fx,cylinder} = \sqrt{2\times\frac{2}{3}g\sin\theta\times d} = \sqrt{\frac{4}{3}gd\sin\theta} = 1.15v_{hoop} \\ v_{fx,ball} = \sqrt{2\times\frac{5}{7}g\sin\theta\times d} = \sqrt{\frac{10}{7}gd\sin\theta} = 1.2v_{hoop} \end{cases}$$



Rotational Motion

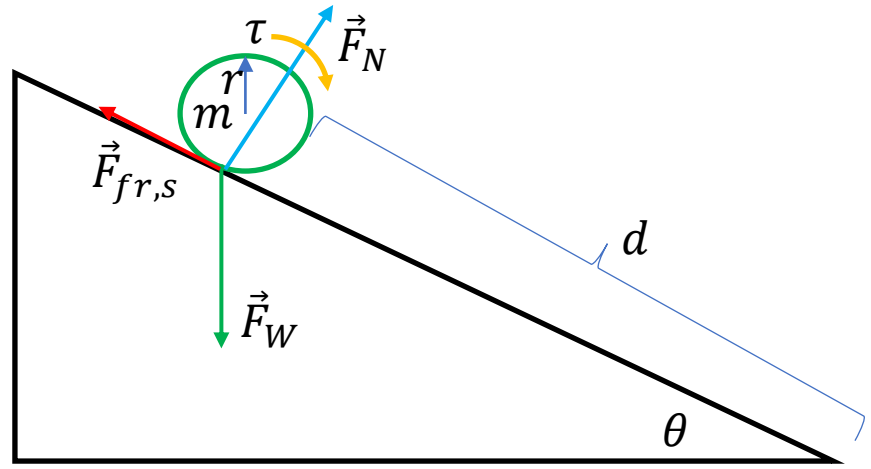
Example 7 Solution continued:

- A couple of comments:

- The speeds do no depend on the size or the mass of the object. I could have said at the start that all of the objects have different mases and radii, but it would have been hard to see that mass and radius has no effect on the speed.

- This was a hard calculation to determine the speeds of the objects, in that we have to calculate the torque and the acceleration. What would we get if we used conservation of energy instead to calculate the speeds of the objects at the bottom of the incline?
- The object with the smallest moment of inertia makes it to the bottom of the incline first since the acceleration of the smaller moment of inertia object is greater.

- Since the moment of inertia is lower the corresponding angular acceleration is greater and the change in velocity is greater.



Rotational Motion

Example 7 Solution continued:

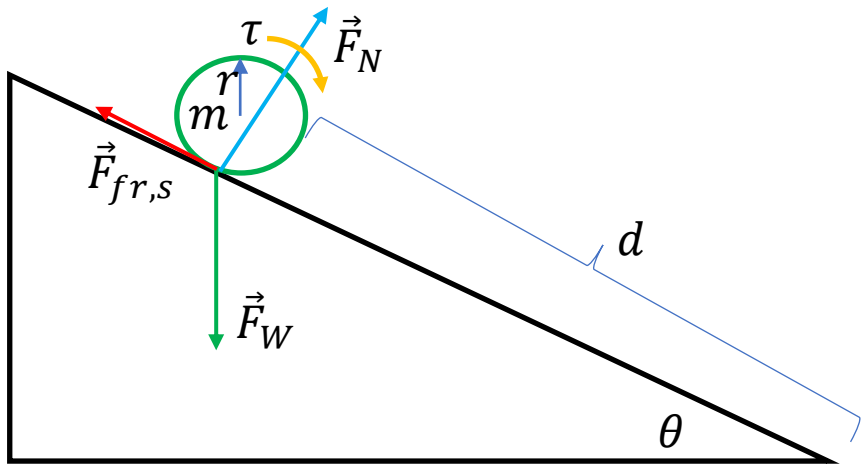
- Since we have the moments of inertia of each object and expressions for the acceleration of the objects, let's calculate the static frictional force and see what they look like.

$$F_{fr,s} = mg\sin\theta - ma_x = mg\sin\theta - m\left(\frac{g\sin\theta}{1+\frac{I}{mr^2}}\right)$$

$$F_{fr,s,hoop} = mg\sin\theta - ma_x = mg\sin\theta - m\left(\frac{g\sin\theta}{1+\frac{mr^2}{mr^2}}\right) = \frac{1}{2}mg\sin\theta$$

$$F_{fr,s,cylinder} = mg\sin\theta - ma_x = mg\sin\theta - m\left(\frac{g\sin\theta}{1+\frac{\frac{1}{2}mr^2}{mr^2}}\right) = \frac{1}{3}mg\sin\theta$$

$$F_{fr,s,ball} = mg\sin\theta - ma_x = mg\sin\theta - m\left(\frac{g\sin\theta}{1+\frac{\frac{2}{5}mr^2}{mr^2}}\right) = \frac{2}{7}mg\sin\theta$$

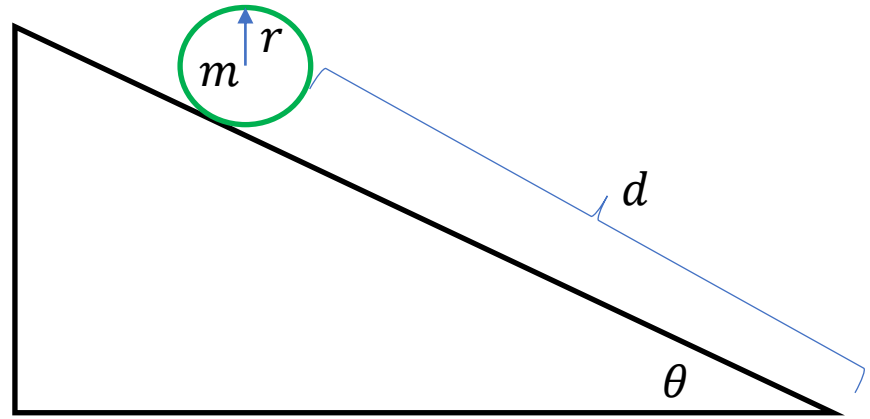


- They don't exactly look like the products of a coefficient of friction and the normal force…. Maybe it's good we didn't write them in that way.

Rotational Dynamics
- Conservation of Energy

Example 7 Solution:

- When the mass m is released from rest rolls down the ramp and falls through a height $\Delta y = y_f - y_i$ from rest translationally ($v_i = 0$) and rotationally ($\omega_i = 0$).



- In addition, each mass spins about an axis through its center and we'll need the moments of inertia for each distribution of mass. From figure 8.20 in the text, $I_{Ring} = mr^2$, $I_{cylinder} = \frac{1}{2}mr^2$, and $I_{ball} = \frac{2}{5}mr^2$.

- Applying conservation of energy we have, where the rotational speed of each mass is related to the rotational speed of the distribution of mass by $v_f = r\omega_f$,

$$\Delta E = \Delta K_R + \Delta K_T + \Delta U_g + \Delta U_s^{0} = 0$$

$$\left(\frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^{2\,0}\right) + \left(\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^{2\,0}\right) + \left(mgy_f^{\,0} - mgy_i\right) = 0$$

m gains rotational K

m gains translational K

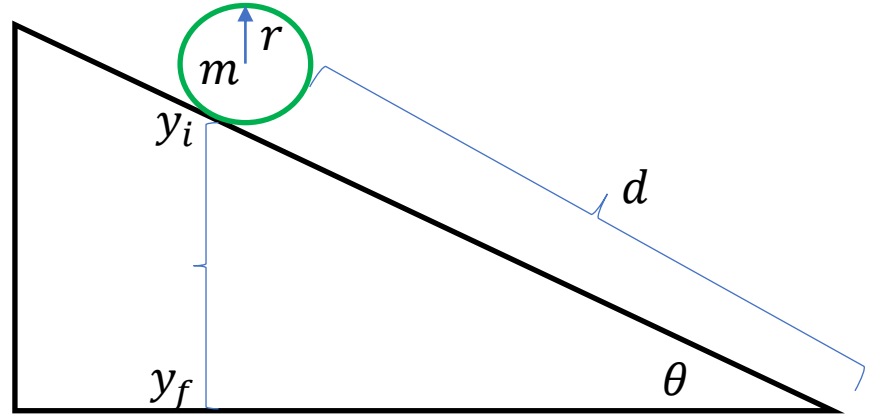
m loses U_g

Rotational Dynamics
- Conservation of Energy

Example 7 Solution continued:

- Our statement of conservation of energy looks like:

$$\underbrace{\frac{1}{2}I\omega_f^2}_{\substack{m \text{ gains} \\ \text{rotational K}}} + \underbrace{\frac{1}{2}mv_f^2}_{\substack{m \text{ gains} \\ \text{translational K}}} \underbrace{- mgy_i}_{m \text{ loses } U_g} = 0$$



- Inserting the expression relating the rotational speed of each object about its center-of-mass and the translational speed.

$$\frac{1}{2}I\left(\frac{v_f}{r}\right)^2 + \frac{1}{2}mv_f^2 - mgy_i = \frac{1}{2}\left(\frac{I}{mr^2} + 1\right)v_f^2 - gy_i = 0$$

- We need to relate the height that the masses fall through to the distance traveled along the ramp. We do this with the geometry in the problem.

$$\sin\theta = \frac{y_i}{d} \rightarrow y_i = d\sin\theta$$

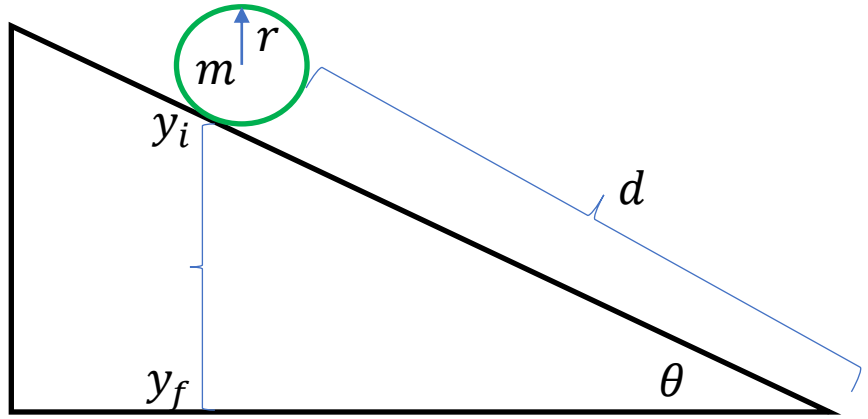
Rotational Dynamics
- Conservation of Energy

Example 7 Solution continued:

- Solving for the final speed of each mass at the bottom of the ramp.

$$\frac{1}{2}\left(\frac{I}{mr^2}+1\right)v_f^2 - gy = \frac{1}{2}\left(\frac{I}{mr^2}+1\right)v_f^2 - gd\sin\theta = 0$$

$$\rightarrow \quad v_f = \sqrt{\frac{2gd\sin\theta}{\frac{I}{mr^2}+1}}$$



- Using each moment of inertia for each distribution of mass we can determine expressions for the speeds of each object at the bottom of the ramp.

$$v_{f,hoop} = \sqrt{\frac{2gd\sin\theta}{\frac{mr^2}{mr^2}+1}} = \sqrt{gd\sin\theta}$$

$$v_{f,cylinder} = \sqrt{\frac{2gd\sin\theta}{\frac{\frac{1}{2}mr^2}{mr^2}+1}} = \sqrt{\frac{4}{3}gd\sin\theta} = 1.15v_{f,hoop}$$

$$v_{f,ball} = \sqrt{\frac{2gd\sin\theta}{\frac{\frac{2}{5}mr^2}{mr^2}+1}} = \sqrt{\frac{10}{7}gd\sin\theta} = 1.20v_{f,hoop}$$

The ball gets to the bottom of the ramp first, followed by the cylinder and lastly the hoop.

These expressions for the speeds are exactly the same as we got using forces/torques.

Rotational Dynamics

Example 8:

The Nott Memorial is a 16-sided Victorian building in the center of campus and is topped with an approximately hemispherical dome of radius R on top of a base that is a height H above the ground.

Suppose that on a windy night someone has precariously placed a pumpkin (of mass m_p and radius r_p, with $r_p \ll R$) at rest atop the Nott. A sudden gust of wind suddenly gets the pumpkin rolling down the dome of the Nott and at some angle θ_R measured with respect to the vertical the pumpkin will lose contact with the dome and become a projectile in flight.

- At what angle θ_R does the pumpkin leave the dome?
- How does this angle compare to the case where the pumpkin just slides down the dome.



https://commodorebuilders.com/2014/03/24/union-college-nott-memorial/

Rotational Dynamics
- Conservation of Energy

Example 8 solution:

- Let's model the Nott Memorial as a square base of height H and a hemispherical roof of radius R. The pumpkin has mass m_p and starts from rest translationally ($v_i = 0$) and rotationally ($\omega_i = 0$). There are no springs in the problem, so let's take $\Delta U_s = 0$.

- To determine the angle θ_R at which the pumpkin leaves the roof, let's start with conservation of energy.

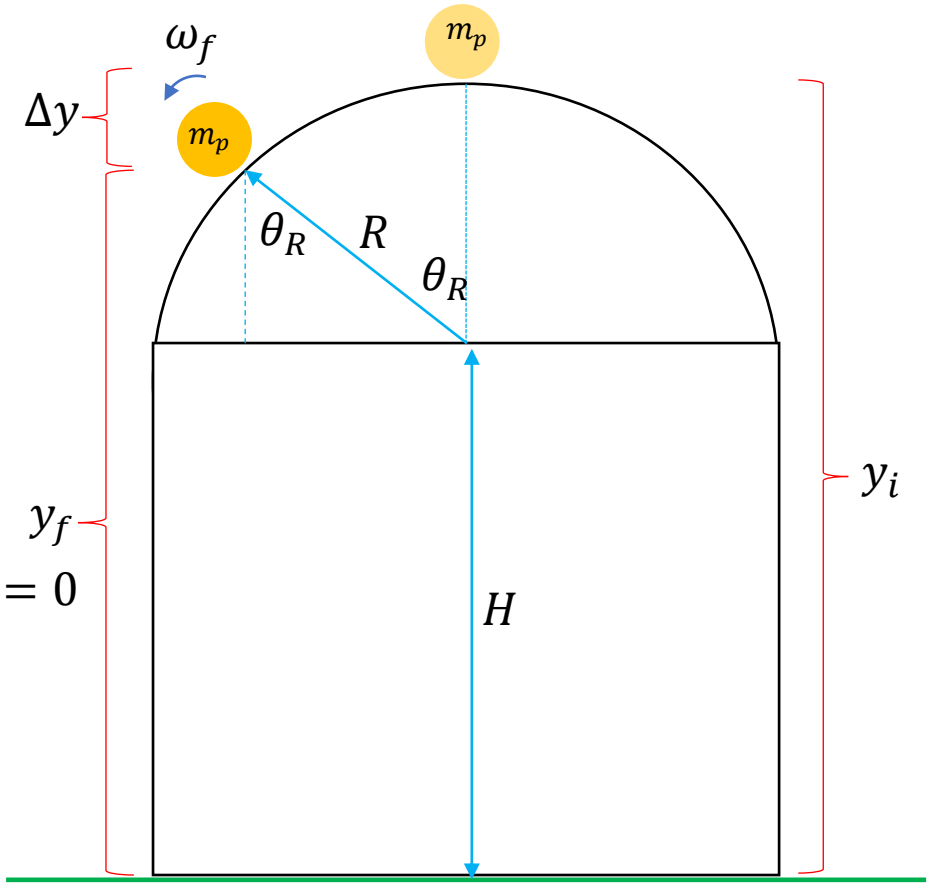
- To get the pumpkin to roll, friction between the pumpkin and the dome of the Nott, exists. If the dome was frictionless, the pumpkin would slide and not roll. That was the previous problem we did.

$$\Delta E = \Delta K_R + \Delta K_T + \Delta U_g + \Delta U_s^{0} = 0$$

$$\Delta E = \left(\frac{1}{2}I_p\omega_f^2 - \frac{1}{2}I_p\omega_i^{2\,0}\right) + \left(\frac{1}{2}m_p v_f^2 - \frac{1}{2}m_p v_i^{2\,0}\right) + \left(m_p g y_f - m_p g y_i\right) = 0$$

$$\Delta E = \frac{1}{2}I_p\omega_f^2 + \frac{1}{2}m_p v_f^2 + \left(m_p g(H + R\cos\theta_R) - m_p g(H + R)\right) = 0$$

These two terms (when multiplied by $m_p g$ subtract and what matters is simply Δy, the height the pumpkin actually fell through.

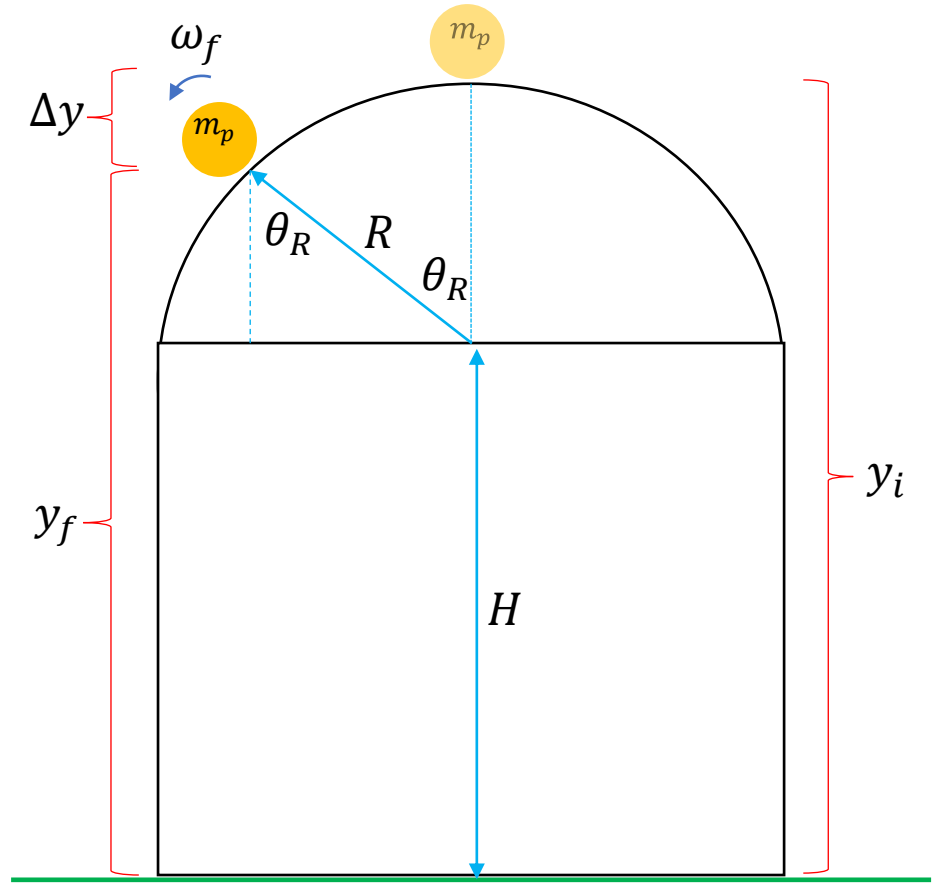


Conservation of Energy

Example 8 solution continued:

$$\Delta E = \frac{1}{2} I_p \omega_f^2 + \frac{1}{2} m_p v_f^2 + m_p g R(\cos\theta_p - 1) = 0$$

- This is the equation we get from conservation of energy.
- It will allow us to solve for θ_R, but unfortunately we have another unknown in the problem.
- That is, what is the translational speed of the pumpkin v_f (as well as the rotational speed ω_f) when it leaves the surface of the dome?
- To determine this, let's look at the motion of the pumpkin.
- The translational and rotational speeds of course is not constant, but are increasing with increasing θ_R. The translational velocity of the pumpkin is changing in magnitude and direction. This is a hard problem to solve.
- However, this looks like a mass m moving in a circle of radius R. So, let's try looking at the forces on the pumpkin just before it leaves the surface of the dome.

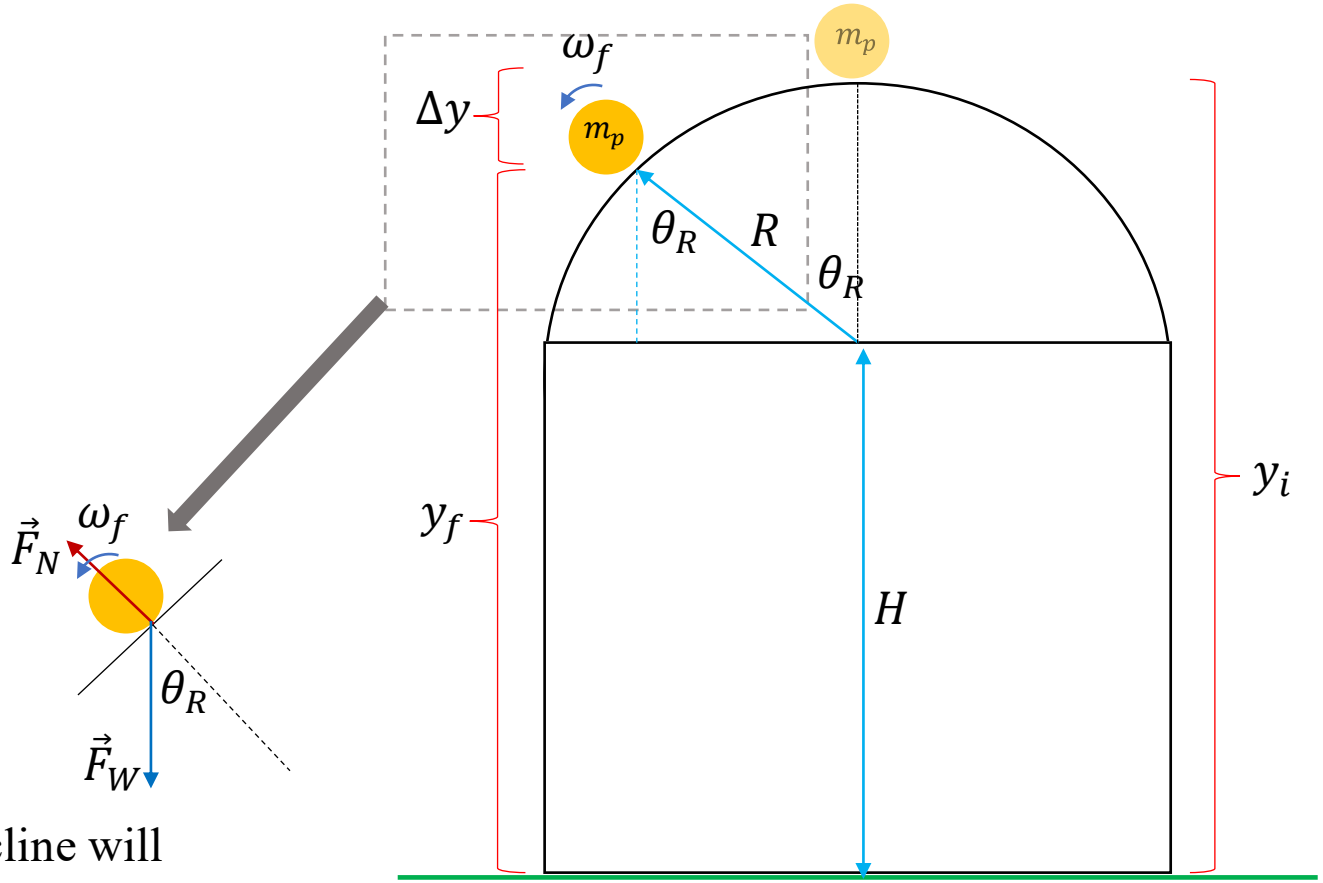


Conservation of Energy

Example 8 solution continued:

- Let's zoom in on a small section of the dome just before the pumpkin leaves.
- This piece of the dome looks locally flat (like a mass on an incline) even though the dome is actually curved.
- Assuming a tilted-coordinate system and applying Newton's laws of motion perpendicular to the incline we have:

$$F_N - F_{Wy} = F_N - m_p g \cos\theta_R = m_p a_y = -m_p \frac{v_f^2}{R}$$



- Applying Newton's laws of motion parallel to the incline will tell us how the speed of the pumpkin is changing. Unfortunately, it's changing with angle, and this is a calculus problem. Let's ignore that as it doesn't help us at all.

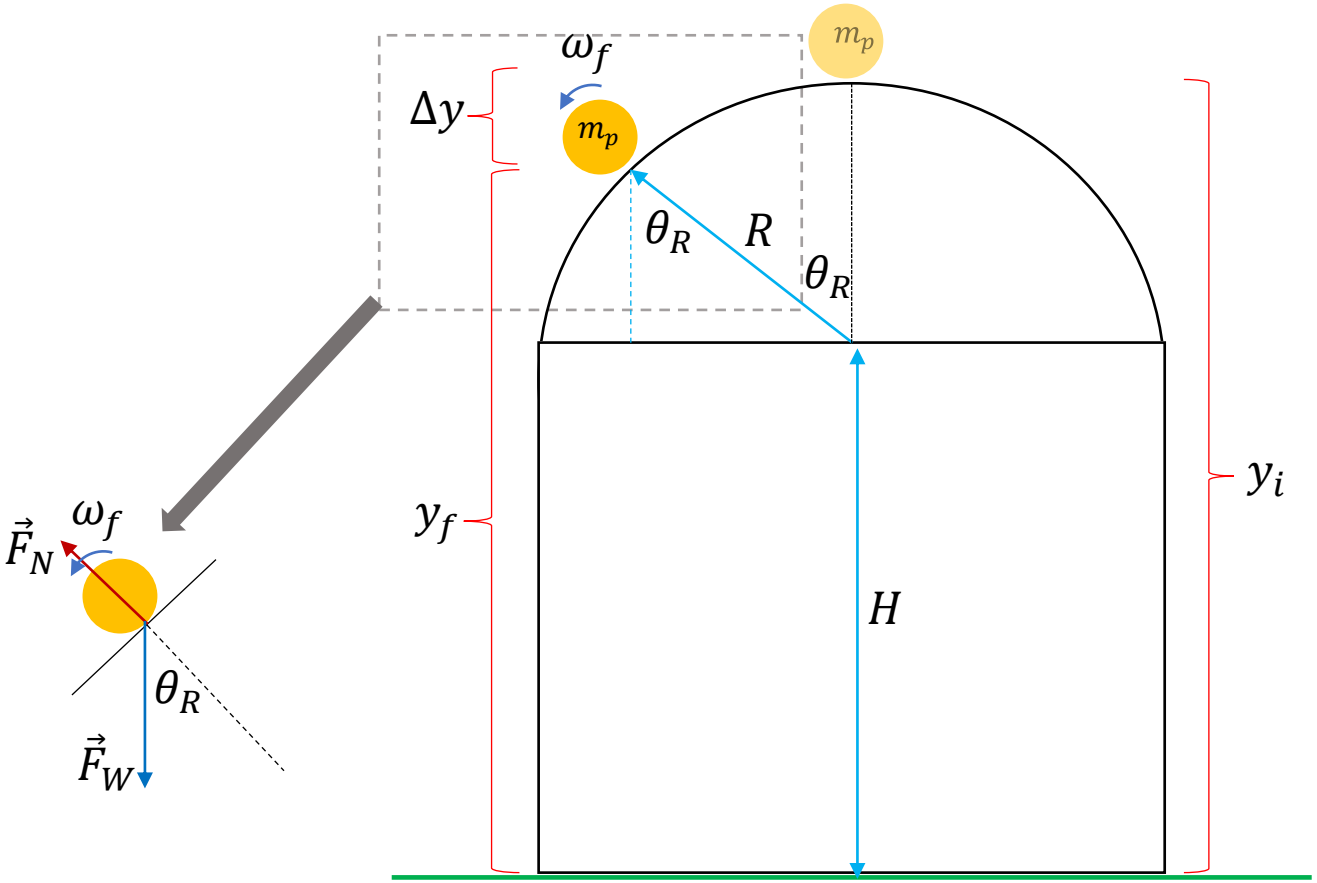
Conservation of Energy

Example 8 solution continued:

- Looking at the motion perpendicular to the dome allows us to relate the speed at which the pumpkin leaves the dome and the angle at which it leaves.
- When the pumpkin leaves the dome, it no longer is in contact with the dome. Thus, $F_N = 0$.

$$\overset{0}{F_N} - m_p g \cos\theta_R = -m_p \frac{v_f^2}{R}$$

$$\boxed{v_f^2 = Rg\cos\theta_R}$$



- Sometimes we need to couple Newton's laws of motion and energy to solve a problem.

Conservation of Energy

Example 8 solution continued:

- Returning to our statement of conservation of energy we can now determine the angle at which the pumpkin leaves the dome using the relation that $v_f = r_p\omega_f$ and the moment of inertia of the pumpkin (modeling it as a sphere) $I_p = \frac{2}{5}m_p r_p^2$.

$$\Delta E = \frac{1}{2}I_p\omega_f^2 + \frac{1}{2}m_p v_f^2 + m_p gR(\cos\theta_R - 1) = 0$$

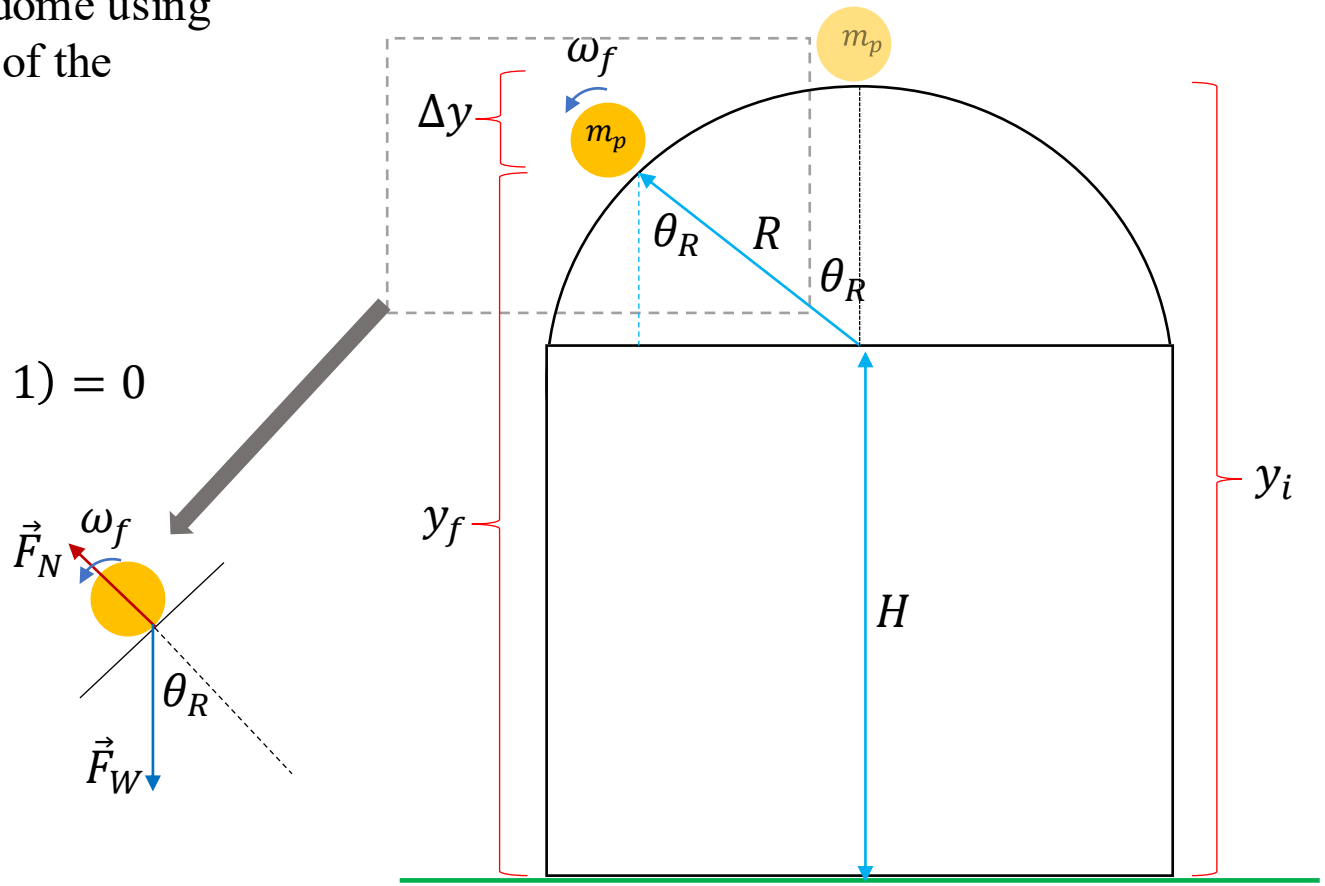
$$\Delta E = \frac{1}{2}\left(\frac{2}{5}m_p r_p^2\right)\left(\frac{v_f}{r_p}\right)^2 + \frac{1}{2}m_p v_f^2 + m_p gR(\cos\theta_R - 1) = 0$$

$$\Delta E = \frac{7}{10}m_p v_f^2 + m_p gR(\cos\theta_R - 1) = 0$$

$$\Delta E = \frac{7}{10}m_p Rg\cos\theta_p + m_p gR(\cos\theta_R - 1) = 0$$

$$\rightarrow \quad \cos\theta_R = \frac{10}{17}$$

$$\rightarrow \quad \theta_R = \cos^{-1}\left(\frac{10}{17}\right) = 54^0$$



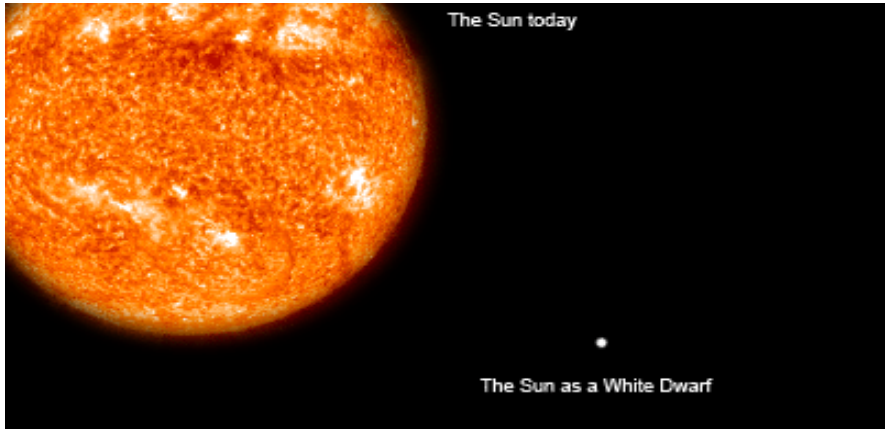
The pumpkin leaves at about 54^0 with respect to the vertical when it rolls. This is greater than when it slides ($\sim 48^0$). The pumpkin needs a certain speed (v_f) to lose contact with the dome. When it slides it get to that speed sooner (all of the loss of U_g goes into K_T). Here when it rolls, the loss of U_g goes into K_T and K_R. Thus it takes longer (leading to a greater angle) to get to v_f.

Rotational Dynamics

Example 9

Stars create energy and pressure (called radiation pressure) inside of their cores by nuclear fusion. The radiation pressure is used to create a force from the core of the star towards its outside to stop the inward squeeze of the force of gravity. The balance of the radiation pressure and gravity is called hydrostatic equilibrium. When the fuel inside of the core is used up, the star dies. The fate of stars, such as our sun, are determined by their mass. Small mass stars, like the sun, will eventually collapse into what is called a white dwarf (the remnant of the stellar core) and lose their outer atmosphere as a nebula. When the sun dies (runs out of fuel in its core) the core collapses and a white dwarf will be formed with the radius of the white dwarf about 1% of the current value of the solar radius (R_s). At the end of its life, the sun will expel its outer atmosphere and collapse into something called a white dwarf. As the sun collapses into a white dwarf, work is done on the collapsing star by the force of gravity. We'll model the sun and the white dwarf as being spheres (balls) spinning about any axis through their centers so that their moments of inertia can be given by $I = \frac{2}{5}mr^2$. Consider the mass of the sun to be $m_s = 2\times10^{30}kg$ and its radius $r_s = 6.96\times10^{8}m$. It takes the sun about 25 days to make one complete revolution on its axis.

- What is the initial rotational speed of the sun?
- What is the final rotational speed of the white dwarf after its collapse? Assume the sun ejects one half of its initial mass, and the rest stays in the white dwarf and that the white dwarf has a radius $r_{wd} = 0.1r_s$.
- What is the change in rotational kinetic energy of the star?
- How much work was done on the star by gravity?

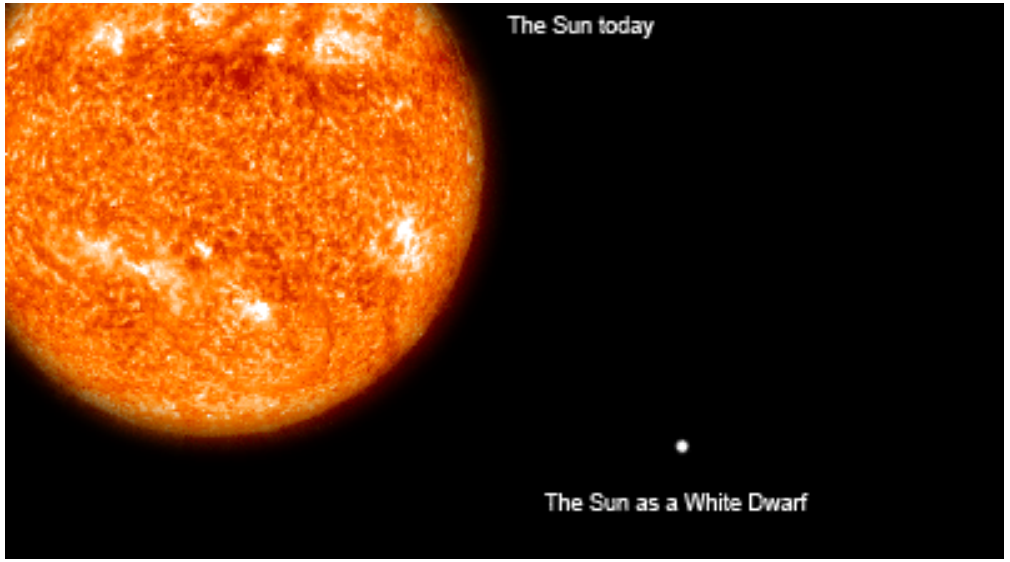


http://migall.fastmail.fm.user.fm/astronomy/stars_and_nebulae/Star_Life_Cycles/page06_wd.htm

Rotational Dynamics

Example 9 Solution:

- We'll model the sun and the white dwarf as being spheres (balls) spinning about any axis through their centers so their moments of inertia will be given by $I = \frac{2}{5}mr^2$. Consider the mass of the sun to be $m_s = 2\times10^{30}kg$ and its radius $r_s = 6.96\times10^{8}m$.



http://migall.fastmail.fm.user.fm/astronomy/stars_and_nebulae/Star_Life_Cycles/page06_wd.htm

- The initial angular momentum of the sun:

$$I_s = \frac{2}{5}m_s r_s^2 = \frac{2}{5}(2\times10^{30}kg)(6.96\times10^{8}m)^2 = 3.58\times10^{47}kgm^2$$

- The initial angular speed of the sun:

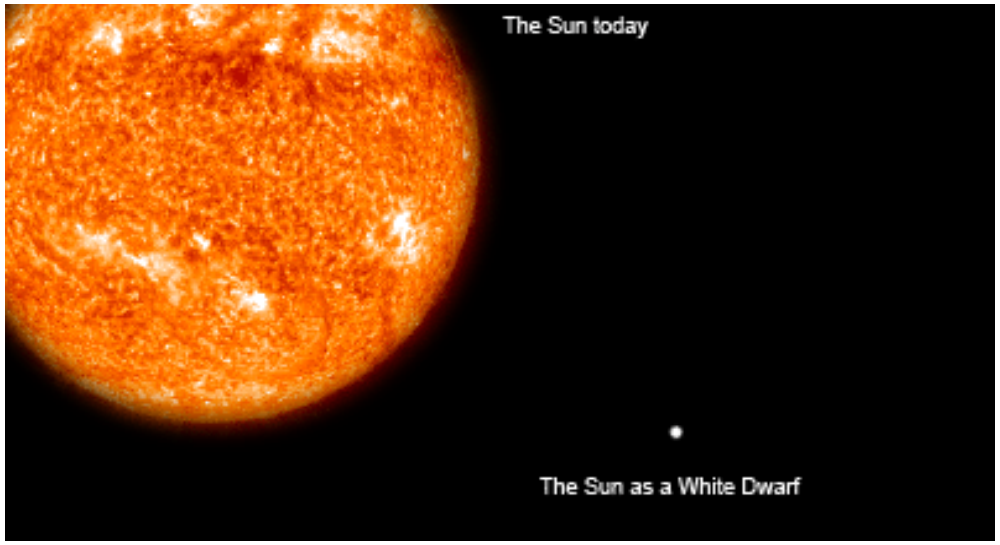
$$\omega_{i,s} = \frac{\Delta\theta}{\Delta t} = \frac{1rev}{30days}\times\frac{1day}{86400s}\times\frac{2\pi rad}{1rev} = 2.4\times10^{-6}\frac{rad}{s}$$

- The initial angular momentum of the sun:

$$L_{i,s} = I_s\omega_{i,s} = 3.58\times10^{47}kgm^2\times2.4\times10^{-6}\frac{rad}{s} = 8.68\times10^{41}\frac{kgm^2}{s}$$

Rotational Dynamics

Example 9 Solution continued:



http://migall.fastmail.fm.user.fm/astronomy/stars_and_nebulae/Star_Life_Cycles/page06_wd.htm

- The angular momentum of the white dwarf assuming that the white dwarf loses half of its initial mass in the collapse.
- Thus, $m_{wd} = 0.5m_s$ and $r_{wd} = 0.01r_s$ from the problem statement

$$I_{wd} = \frac{2}{5}m_{wd}r_{wd}^2 = \frac{2}{5}(0.5\times2\times10^{30}kg)(0.01\times6.96\times10^{8}m)^2 = 1.94\times10^{43}kgm^2$$

- The final angular momentum of the white dwarf by conservation of angular momentum:

$$L_{f,wd} = L_{i,s}$$

- The angular speed of the white dwarf:

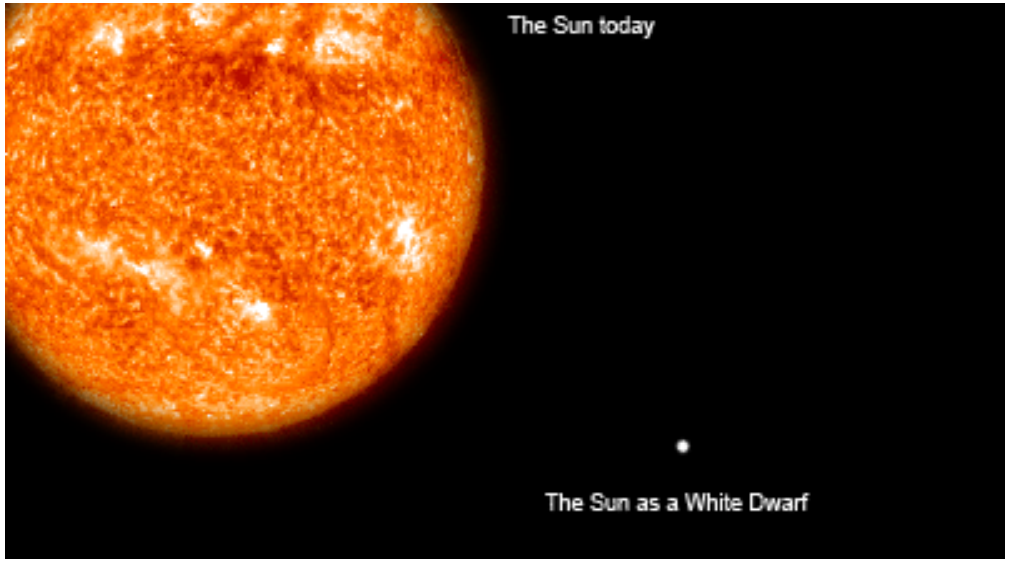
$$L_{f,wd} = L_{i,s} \rightarrow I_{wd}\omega_{wd} = I_s\omega_s \rightarrow \omega_{wd} = \frac{I_s}{I_{wd}}\omega_s = \frac{3.58\times10^{47}kgm^2}{1.94\times10^{43}kgm^2}\times2.4\times10^{-6}\frac{rad}{s} = 0.045\frac{rad}{s}$$

The final rotational speed of the white dwarf is about 20000 times greater than the rotational speed of the sun. The white dwarf spins very fast.

Rotational Kinetic Energy

Example 9 Solution continued:

As the sun collapses into a white dwarf how much work is done by the force of gravity on the collapsing star?



http://migall.fastmail.fm.user.fm/astronomy/stars_and_nebulae/Star_Life_Cycles/page06_wd.htm

- The work done by gravity on the collapsing star is given by the change in rotational kinetic energies of the system:

$$K_{Ri} = \frac{1}{2} I_s \omega_{s,i}^2 = \frac{1}{2}(3.58\times10^{47} kgm^2)\left(2.4\times10^{-6}\frac{rad}{s}\right)^2 = 1.03104\times10^{36} J$$

$$K_{Rf} = \frac{1}{2} I_{wd} \omega_{wd,f}^2 = \frac{1}{2}(1.94\times10^{43} kgm^2)\left(0.045\frac{rad}{s}\right)^2 = 1.96425\times10^{40} J$$

- Rotational work is done and the work done is:

$$W_R = K_{R,f} - K_{R,i} = 1.96415\times10^{40} J$$

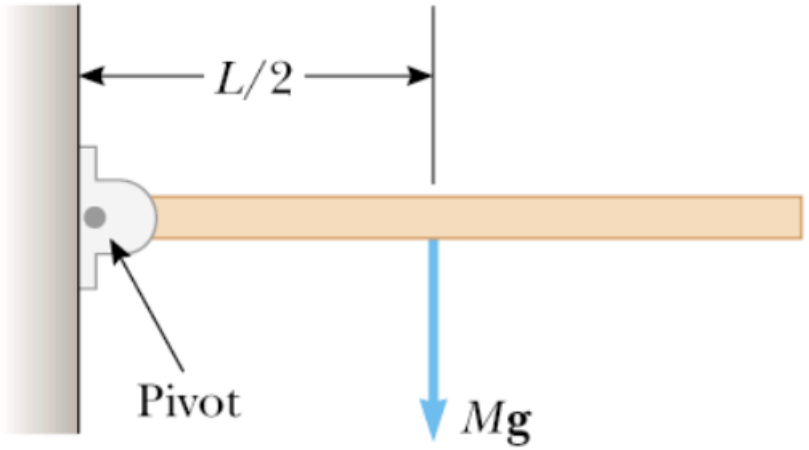
Rotational work was done by the force of gravity pulling the collapsing star into a smaller radius. The force of gravity acts toward the center of the star and the motion of the star is towards its center (getting smaller). A positive amount of work is done on the star, and its rotational kinetic energy therefore increases. Here the initial rotational kinetic energy is large, but the final is 4 orders of magnitude larger! This white dwarf has an amazing amount of energy associated with it.

Rotational Dynamics

Example 10:

A uniform rod of mass M and length L is attached to a pivot on one end and is held horizontally at the free end. The rod is released from rest and allowed to fall.

- What is the initial angular acceleration of the rod?
- What is the translational acceleration of the pivot, the center-of-mass of the rod, and the tip of the rod farthest from the pivot.
- At what distance from the pivot is the translational acceleration equal to g?



Rotational Motion

Example 10 Solution:

- To calculate the initial angular acceleration of the rod, we calculate the torque on the rod. The rod rotates clockwise, and the direction of the torque is negative and the magnitude of the torque

$$\tau = rF\sin\theta = \frac{L}{2}Mg\sin 90 = \frac{MgL}{2}$$

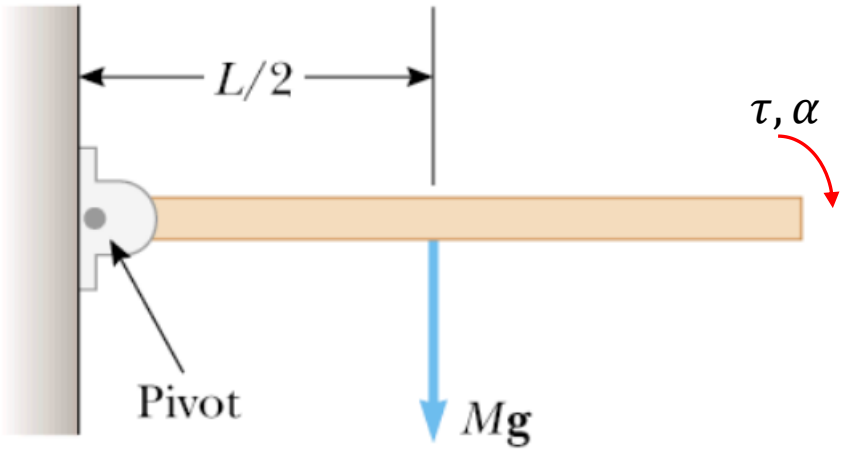
- The torque is related to the moment of inertia of the rod and the angular acceleration of rod by Newton's second law.

- From Table 7.1 on page 226 of KJF College Physics: A Strategic Approach, Volume 1, for a uniform rod spun about one end, the moment of inertia is $I = \frac{1}{3}ML^2$.

- Thus, the initial angular acceleration of the rod is:

$$\tau = I\alpha \rightarrow \alpha = \frac{\tau}{I} = \frac{3MgL}{2ML^2} = \frac{3g}{2L}$$

- This is the acceleration of the rod about the axis of rotation (the pivot). The whole rod experiences this angular acceleration as the rod has to remain rigid.



Note: Once the rod starts to move, the angle between the weight and the moment arm changes. When the rod is vertical the angle between the weight of the rod and the moment arm is 0^0. Thus, the torque changes with angle from a maximum when the rod is horizontal to a minimum when the rod is vertical. The torque is an angular dependent quantity here. So, we *could not* use the equations of motion for constant angular acceleration to describe the motion. We could use energy though.

Rotational Motion

Example 10 Solution continued:

- Having the angular acceleration of the system we can calculate the translational acceleration of various parts of the rod.
- Since the rod has to remain rigid, different parts must experience different magnitudes of the translational acceleration. If the translational acceleration of each part was the same the rod would snap.
- The translational acceleration of the pivot:

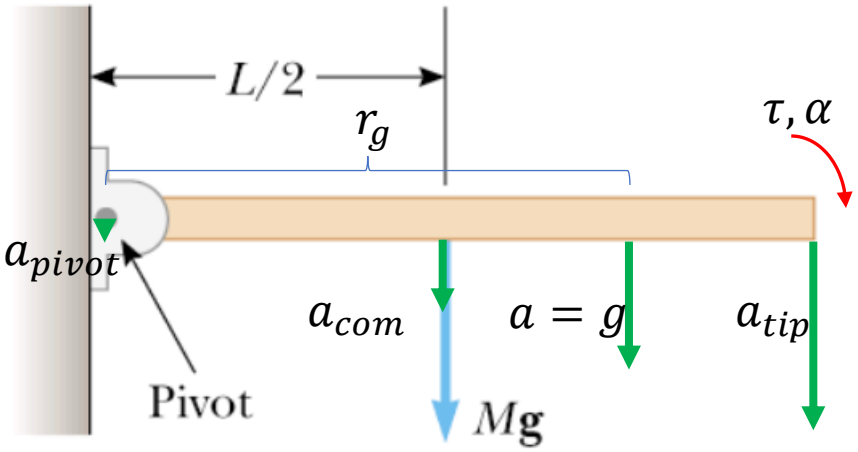
$$a_{pivot} = r_{pivot}\alpha_{pivot} = (0)\alpha = 0$$

- The translational acceleration of the center-of-mass of the rod:

$$a_{com} = r_{com}\alpha_{com} = \left(\frac{L}{2}\right)\alpha = \frac{L}{2}\times\frac{3g}{2L} = \frac{3}{4}g$$

- The translational acceleration of the tip of the rod:

$$a_{tip} = r_{tip}\alpha_{tip} = (L)\alpha = L\times\frac{3g}{2L} = \frac{3}{2}g$$



- The distance from the pivot to where the translational acceleration is g:

$$a_g = r_g\alpha_g \rightarrow r_g = \frac{g}{\alpha} = \frac{g2L}{3g} = \frac{2}{3}L$$