Physics 110
Spring 2006
Springs – Their Solutions

1. When a 4.0kg mass is hung vertically on a light spring that obeys Hooke's law, the spring stretches 2.5cm.
   a. If the 4.0kg mass is removed, how far will the spring stretch if a 1.5kg mass is hung from the same spring?
   b. How much work must an external agent do to stretch the spring 4.0cm from its un-stretched length?

   a. The spring constant is obtained from Hooke's law $k = \frac{\Delta F}{\Delta x} = \frac{mg}{x} = \frac{4kg \times 9.8\frac{m}{s^2}}{0.025m} = 1568\frac{N}{m}$. For the new mass, the stretch is given as $x = \frac{\Delta F}{k} = \frac{mg}{x} = \frac{1.5kg \times 9.8\frac{m}{s^2}}{1568\frac{N}{m}} = 0.0094m = 0.94cm = 9.4mm$.
   b. The work is $W = \frac{1}{2}kx^2 = \frac{1}{2} \times 1568\frac{N}{m} \times (0.04m)^2 = 1.25J$.

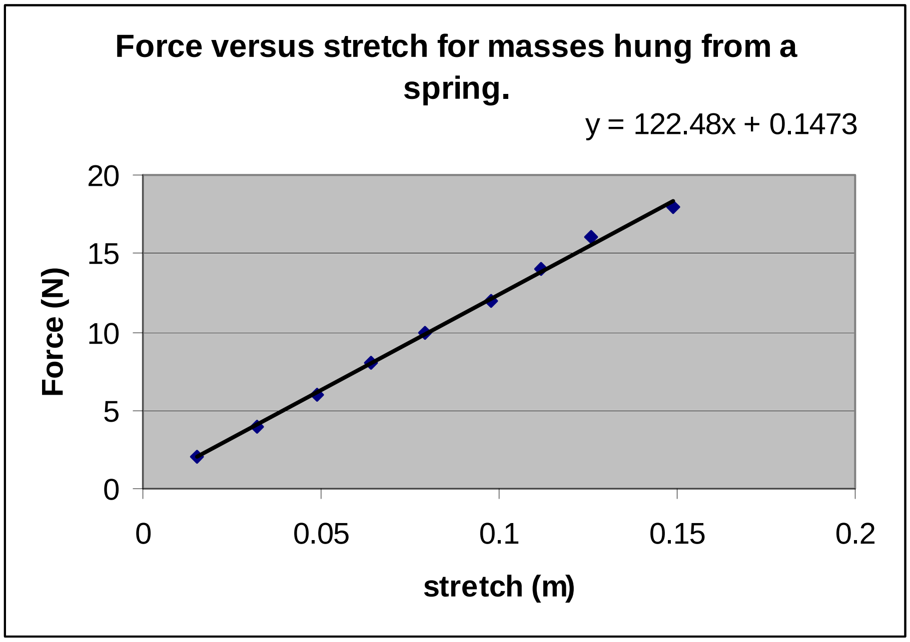
2. If it takes 4.0J of work to stretch a Hooke's law spring 10.0cm from its un-stretched length, how much extra work is required to stretch it an additional 10.0cm?

   From the work given we find the spring constant $k = \frac{2W}{x^2} = \frac{2 \times 4J}{(0.01m)^2} = 800\frac{N}{m}$.
   Then the work required to stretch the spring by 20cm, is $W = \frac{1}{2}kx^2 = \frac{1}{2} \times 800\frac{N}{m} \times (0.20m)^2 = 16J$. Therefore the extra work required to stretch the spring by 10cm is 16J – 4J = 12J.

3. When different weights are hung on a spring, the spring stretches to different lengths as shown in the table below.
   a. Make a graph of the applied force versus the stretch of the spring and if the data are linear obtain the slope of the best fit line. What does this slope represent?
   b. If the spring is stretched 105cm, what force dies the spring exert on the suspended weight?

F (N)	2	4	6	8	10	12	14	16	18
x (mm)	15	32	49	64	79	98	112	126	149

a. The data were graphed on Excel and the result is plotted below.



From the graph above, the spring is fairly linear (except at the very largest stretches perhaps.) Thus the slope represents the spring constant and has a value of 122.5 N/m.

b. The force require to stretch the spring by 105mm is obtained from Hooke's law and has a value of 12.9N.

4. A 200g block is pressed against a spring with spring constant 1.4kN/m until the block compresses the spring 10cm. The spring rests at the bottom of a ramp inclined at 60^{o} to the horizontal. Using energy methods how far up the incline does the block move before it stops if there is no friction present and if there is friction present where the coefficient of kinetic friction is 0.400?

No Friction:

Equating the energy at the bottom of the ramp to the energy at a distance d up the ramp (at a height h in the vertical direction) we find

$$\tfrac{1}{2}kx^2 = \tfrac{1}{2}\times 1.4\times 10^3\,\tfrac{N}{m}\times(0.1m)^2 = 7J$$

$$7J = mgh = mgd\sin\theta \rightarrow d = \frac{7J}{0.2kg\times 9.8\frac{m}{s^2}\times\sin 60} = 4.12m$$

Friction:

Here we equate the initial energy, still 7J, at the bottom of the ramp to the total energy at a distance d' up the ramp (at a height h') which is the sum of the gravitational potential energy (at h') and the energy losses due to friction. The energy losses due to friction are calculated from the work done by friction which is the force of friction times the displacement of the object. The force of friction is simply the product of the normal force and the coefficient of kinetic friction. The normal force is found through a free body diagram and has magnitude $F_N = mg\cos\theta$. Putting this all together we have

$$7J = mgh' + \Delta E_{friction} = mgd'\sin\theta + \mu_k mgd'\cos\theta$$

$$\rightarrow d' = \frac{7J}{\left(0.2kg \times 9.8\,\tfrac{m}{s^2} \times \sin 60\right)\times\left[1 + 0.4 \times \frac{\cos 60}{\sin 60}\right]} = 3.35m$$

5. A 0.5kg mass is attached to a spring with a spring constant of 8.0N/m and vibrates with an amplitude of 10cm.
    a. What are the maximum values for the magnitudes of the speed and of the acceleration?
    b. What are the speed and the acceleration when the mass is 6cm from the equilibrium position?
    c. What is the time it takes the mass to move from x = 0cm to x = 8cm?
    d. What is the period of the motion?
    e. What are the displacement, velocity and acceleration as functions of time?

a. The maximum speed and acceleration are given by respectively by

$$v_{\max} = \sqrt{\frac{k}{m}}A = \sqrt{\frac{8\,\frac{N}{m}}{0.5kg}} \times 0.10m = 0.4\,\tfrac{m}{s}$$

$$a_{\max} = \frac{k}{m}A = \frac{8\,\frac{N}{m}}{0.5kg} \times 0.10m = 1.6\,\tfrac{m}{s^2}.$$

b. To calculate the velocity at x = 6cm, we use $v = v_{\max}\left(1 - \frac{x^2}{A^2}\right)^{\frac{1}{2}} = 0.4\,\tfrac{m}{s}\left(1 - \frac{(0.06m)^2}{(0.10m)^2}\right)^{\frac{1}{2}} = 0.32\,\tfrac{m}{s}$. To calculate the acceleration we can use one of two methods. One uses calculus and one does not. Using no calculus, we take the trajectory equation and solve for t and then use this time in the equation of motion for the acceleration. (Note the period is found in part d below.)

$$x(t) = A\sin\left(\frac{2\pi t}{T}\right) \rightarrow 0.06m = 0.10m\sin\left(\frac{2\pi t}{1.57s}\right) \rightarrow t = 0.161s$$

$$a(t) = -\frac{k}{m}A\sin\left(\frac{2\pi t}{T}\right) = -\frac{8\,\frac{N}{m}}{0.5kg} \times 0.10m \times \sin\left(\frac{2\pi \times 0.161s}{1.57s}\right) = -0.96\,\tfrac{m}{s^2}.$$

Using calculus, we take the derivative of the above expression with respect to time. Using the chain rule we find that

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v = \frac{-v_{\max}x}{A^2\left(1 - \frac{x^2}{A^2}\right)^{\frac{1}{2}}}v = -\frac{0.4\,\frac{m}{s} \times 0.06m \times 0.32\,\frac{m}{s}}{(0.10m)^2\left(1 - \frac{(0.06m)^2}{(0.10m)^2}\right)^{\frac{1}{2}}} = -0.96\,\tfrac{m}{s^2}$$

c. Using the trajectory equation

$t_0 = 0s$

$$t_8 = \frac{T}{2\pi}\sin^{-1}\left(\frac{x}{A}\right) = \frac{1.57s}{2\pi}\sin^{-1}\left(\frac{0.08m}{0.10m}\right) = 0.232s$$

$\Delta t = t_8 - t_0 = 0.232s$

d. The period is obtained from $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.5kg}{8\frac{N}{m}}} = 1.57s$.

e. The equations of motion are thus

$$x(t) = 0.1m\sin\left(4s^{-1}t\right)$$
$$v(t) = 0.4\tfrac{m}{s}\cos\left(4s^{-1}t\right)$$
$$a(t) = -1.6\tfrac{m}{s^2}\sin\left(4s^{-1}t\right)$$

6. A 1.50kg block at rest on a horizontal tabletop is attached to a horizontal spring having a force constant of 19.6N/m. The spring is initially un-stretched. A constant 20.0N horizontal force is applied to the block causing the spring to stretch.
   a. What is the speed of the mass after it has moved 0.3m from the equilibrium position if the surface that the block is resting on is frictionless.
   b. What is the speed of the mass after it has moved 0.3m from the equilibrium position if the surface that the block resting on is not frictionless but has a coefficient of kinetic friction of 0.2?

a.

$$W = \Delta E = 20N \times 0.3m = 6J$$

$$6J = \tfrac{1}{2}mv_f^2 + \tfrac{1}{2}kx^2 = 0.75v_f^2 + \tfrac{1}{2}\times 19.6\tfrac{N}{m}\times(0.3m)^2 \rightarrow v_f = 2.61\tfrac{m}{s}$$

b.

$$W_{me} = \left(F_{app} - k\Delta x\right)\Delta x = \left(20N - 19.6\tfrac{N}{m}\times 0.3m\right)\times 0.3m = 4.24J$$

$$4.24J = \tfrac{1}{2}mv_{f,new}^2 \rightarrow v_{f,new} = 2.38\tfrac{m}{s}$$

7. After a thrilling plunge bungee jumpers bounce freely on the bungee cords though many cycles. After watching many people of differing masses jump and then bounce up and down, you come up with a relation: A mass *m* is oscillating freely on a vertical spring with a period *T*. An unknown mass *m'* on the same spring oscillates with a period *T'*.
   a. What is the spring constant of the bungee cord?
   b. What is the unknown mass *m'*?

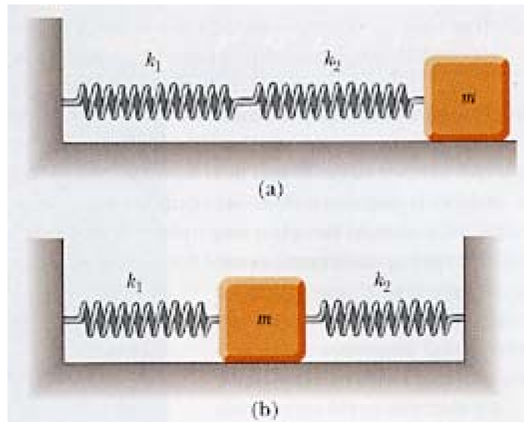
a. The spring constant is obtained from the period-frequency relation and is given by $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \rightarrow k = 4\pi^2 mf^2 = \frac{4\pi^2 m}{T^2}$.

b. Since each person uses the same bungee cord, the spring constants are the same, so thus we have $k = \frac{4\pi^2 m}{T^2} = \frac{4\pi^2 m'}{(T')^2} \rightarrow m' = m\frac{(T')^2}{T^2}$.

8. A mass is connected to two springs of force constants k_1 and k_2 as shown below. In each case the mass moves on a frictionless table and is displaced from its equilibrium position and then released. Show that in each case the mass exhibits simple harmonic motion with periods

   a. $T = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$

   b. $T = 2\pi\sqrt{\frac{m}{(k_1 + k_2)}}$



a. For this case as I pull the mass to the right, say, an amount x from its equilibrium position, each mass stretches by some amount. Let the total amount be $x = x_1 + x_2$. From Hooke's law we have

$$x = x_1 + x_2 = \frac{F_{applied}}{k_1} + \frac{F_{applied}}{k_2} = \frac{k_1 + k_2}{k_1 k_2} F_{applied} \rightarrow k_{eff} = \frac{k_1 k_2}{k_1 + k_2}.$$

$$T = 2\pi\sqrt{\frac{m}{k_{eff}}} = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$

b. If I pull the mass to the right, say, in this case, spring 2 compresses by x and spring 1 extends by x. Thus

$$F_{applied} = -k_1 x - k_2 x = -k_{eff} x \rightarrow k_{eff} = k_1 + k_2$$

$$T = 2\pi\sqrt{\frac{m}{k_{eff}}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$