Physics 111
Fall 2007
Wave Optics Solutions

1. The pupil of a cat's eye narrows to a vertical slit of width 0.500 mm in daylight. What is the angular resolution for horizontally separated mice? Assume that the average wavelength of the light is 500 nm.

$$\sin \theta = \frac{\lambda}{a} = \frac{5.00 \times 10^{-7} \text{ m}}{5.00 \times 10^{-4}} = \boxed{1.00 \times 10^{-3} \text{ rad}}$$

2. Many cells are transparent and colorless. Structures of great interest in biology and medicine can be practically invisible to ordinary microscopy. An *interference microscope* reveals a difference in refractive index as a shift in interference fringes to indicate the size and shape of cell structures. The idea is exemplified in the following problem. An air wedge is formed between two glass plates in contact along one edge and slightly separated at the opposite edge. When the plates are illuminated with monochromatic light from above, the reflected light has 85 dark fringes. Calculate the number of dark fringes that appear if water ($n = 1.33$) replaces the air between the plates.

    For dark fringes, $2nt = m\lambda$ and at the
    edge of the wedge, $t = \dfrac{84(500 \text{ nm})}{2}$.
    When submerged in water, $2nt = m\lambda$

    $$m = \frac{2(1.33)(42)(500 \text{ nm})}{500 \text{ nm}} \text{ so}$$

    $m + 1 = \boxed{113 \text{ dark fringes}}$.
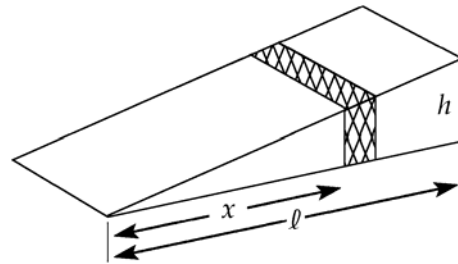
3. The *Very Large Array* (VLA) is a set of 27 radio telescope dishes in Caton and Socorro counties, New Mexico, and one of them is shown below. The antennas can be moved apart on railroad tracks, and their combined signals give the resolving power of a synthetic aperture 36.0 km in diameter.

    (a) If the detectors are tuned to a frequency of 1.40 GHz, what is the angular resolution of the VLA?
    (b) Clouds of hydrogen radiate at this frequency. What must be the separation distance of two clouds at the center of the galaxy, 26 000 lightyears away, if they are to be resolved?

(c) As the telescope looks up, a circling hawk looks down. Find the angular resolution of the hawk's eye. Assume that that the hawk is most sensitive to green light having wavelength 500 nm and that it has a pupil of diameter 12.0 mm.

(d) A mouse is on the ground 30.0 m below. By what distance must the mouse's whiskers be separated if the hawk can resolve them?

(a) $\lambda = \dfrac{v}{f}$: $\lambda = \dfrac{3.00 \times 10^8 \text{ m/s}}{1.40 \times 10^9 \text{ s}^{-1}} = 0.214 \text{ m}$

$\theta_{\min} = 1.22\dfrac{\lambda}{D}$: $\theta_{\min} = 1.22\left(\dfrac{0.214 \text{ m}}{3.60 \times 10^4 \text{ m}}\right) = \boxed{7.26 \; \mu\text{rad}}$

$\theta_{\min} = 7.26 \; \mu\text{rad}\left(\dfrac{180 \times 60 \times 60 \text{ s}}{\pi}\right) = \boxed{1.50 \text{ arc seconds}}$

(b) $\theta_{\min} = \dfrac{d}{L}$: $d = \theta_{\min}L = \left(7.26 \times 10^{-6} \text{ rad}\right)\left(26\,000 \text{ ly}\right) = \boxed{0.189 \text{ ly}}$

(c) $\theta_{\min} = 1.22\dfrac{\lambda}{D}$

$\theta_{\min} = 1.22\left(\dfrac{500 \times 10^{-9} \text{ m}}{12.0 \times 10^{-3} \text{ m}}\right) = \boxed{50.8 \; \mu\text{rad}} \, (10.5 \text{ seconds of arc})$

(d) $d = \theta_{\min}L = \left(50.8 \times 10^{-6} \text{ rad}\right)\left(30.0 \text{ m}\right) = 1.52 \times 10^{-3} \text{ m} = \boxed{1.52 \text{ mm}}$

4. Iridescent peacock feathers are shown in figure a below. The surface of one microscopic barbule is composed of transparent keratin that supports rods of dark brown melanin in a regular lattice, represented in figure b. (Your fingernails are made of keratin, and melanin is the dark pigment giving color to human skin.) In a portion of the feather that can appear turquoise, assume that the melanin rods are uniformly separated by 0.25 μm, with air between them.
   (a) Explain how this structure can appear blue–green when it contains no blue or green pigment.
   (b) Explain how it can also appear violet if light falls on it in a different direction.
   (c) Explain how it can present different colors to your two eyes at the same time, a characteristic of iridescence.
   (d) A compact disc can appear to be any color of the rainbow. Explain why this portion of the feather cannot appear yellow or red.
   (e) What could be different about the array of melanin rods in a portion of the feather that does appear to be red?

(a)

(a) Bragg's law applies to the space lattice of melanin rods. Consider the planes $d = 0.25 \ \mu m$ apart. For light at near-normal incidence, strong reflection happens for the wavelength given by $2d\sin\theta = m\lambda$. The longest wavelength reflected strongly corresponds to $m = 1$:

$2\left(0.25 \times 10^{-6} \ m\right)\sin 90° = 1\lambda$, $\lambda = 500$ nm. This is the blue-green color.

(b) For light incident at grazing angle 60°, $2d\sin\theta = m\lambda$ gives
$1\lambda = 2\left(0.25 \times 10^{-6} \ m\right)\sin 60° = 433$ nm. This is violet.

(c) Your two eyes receive light reflected from the feather at different angles, so they receive light incident at different angles and containing different colors reinforced by constructive interference.

(d) The longest wavelength that can be reflected with extra strength by these melanin rods is the one we computed first, 500 nm blue-green.

(e) If the melanin rods were farther apart (say $0.32 \ \mu m$) they could reflect red with constructive interference.

5. Electrons are ejected from a metallic surface with speeds ranging up to 4.60×10^5 m/s when light with a wavelength of 625 nm is used.
   (a) What is the work function of the surface?
   (b) What is the cutoff frequency for this surface?

$$K_{max} = \frac{1}{2}mv_{max}^2 = \frac{1}{2}\left(9.11 \times 10^{-31}\right)\left(4.60 \times 10^5\right)^2 = 9.64 \times 10^{-20} \ J = 0.602 \ eV$$

(a) $\phi = E - K_{max} = \dfrac{1 \ 240 \ eV \cdot nm}{625 \ nm} - 0.602 \ nm = \boxed{1.38 \ eV}$

(b) $f_c = \dfrac{\phi}{h} = \dfrac{1.38 \ eV}{6.626 \times 10^{-34} \ J \cdot s}\left(\dfrac{1.60 \times 10^{-19} \ J}{1 \ eV}\right) = \boxed{3.34 \times 10^{14} \ Hz}$

6. X-rays having an energy of 300 keV undergo Compton scattering from a target. The scattered rays are detected at 37.0° relative to the incident rays. Find
   (a) the Compton shift at this angle,

(b) the energy of the scattered x-ray, and

(c) the energy of the recoiling electron.

(a) $\Delta\lambda = \dfrac{h}{m_e c}(1 - \cos\theta)$:

$$\Delta\lambda = \frac{6.626 \times 10^{-34}}{\left(9.11 \times 10^{-31}\right)\left(3.00 \times 10^{8}\right)}(1 - \cos 37.0°) = \boxed{4.88 \times 10^{-13} \text{ m}}$$

(b) $E_0 = \dfrac{hc}{\lambda_0}$:

$$\left(300 \times 10^{3} \text{ eV}\right)\left(1.60 \times 10^{-19} \text{ J/eV}\right) = \frac{\left(6.626 \times 10^{-34}\right)\left(3.00 \times 10^{8} \text{ m/s}\right)}{\lambda_0}$$

$$\lambda_0 = 4.14 \times 10^{-12} \text{ m}$$

and $\lambda' = \lambda_0 + \Delta\lambda = 4.63 \times 10^{-12} \text{ m}$

$$E' = \frac{hc}{\lambda'} = \frac{\left(6.626 \times 10^{-34} \text{ J}\cdot\text{s}\right)\left(3.00 \times 10^{8} \text{ m/s}\right)}{4.63 \times 10^{-12} \text{ m}} = 4.30 \times 10^{-14} \text{ J} = \boxed{268 \text{ keV}}$$

(c) $K_e = E_0 - E' = 300 \text{ keV} - 268.5 \text{ keV} = \boxed{31.5 \text{ keV}}$

7. After learning about de Broglie's hypothesis that particles of momentum p have wave characteristics with wavelength $\lambda = h/p$, an 80.0-kg student has grown concerned about being diffracted when passing through a 75.0-cm-wide doorway. Assume that significant diffraction occurs when the width of the diffraction aperture is less than 10.0 times the wavelength of the wave being diffracted.

   (a) Determine the maximum speed at which the student can pass through the doorway so as to be significantly diffracted.

   (b) With that speed, how long will it take the student to pass through the doorway if it is in a wall 15.0 cm thick? Compare your result to the currently accepted age of the Universe, which is 4×10^{17} s.

   (c) Should this student worry about being diffracted?

(a) The wavelength of the student is $\lambda = \dfrac{h}{p} = \dfrac{h}{mv}$. If w is the width of the diffracting aperture,

then we need $\qquad w \leq 10.0\lambda = 10.0\left(\dfrac{h}{mv}\right)$

so that

$$v \leq 10.0 \frac{h}{mw} = 10.0 \left(\frac{6.626 \times 10^{-34} \ \mathrm{J \cdot s}}{(80.0 \ \mathrm{kg})(0.750 \ \mathrm{m})} \right) = \boxed{1.10 \times 10^{-34} \ \mathrm{m/s}}.$$

(b)   Using $\Delta t = \dfrac{d}{v}$ we get: $\Delta t \geq \dfrac{0.150 \ \mathrm{m}}{1.10 \times 10^{-34} \ \mathrm{m/s}} = \boxed{1.36 \times 10^{33} \ \mathrm{s}}.$

(c) $\boxed{\text{No}}$. The minimum time to pass through the door is over 10^{15} times the age of the Universe.

8. A radio station operating at 106.5 MHz broadcasts from two identical antennae at same elevation but separated by an 8.0-m horizontal distance d, as shown below. A maximum signal is found along the midline, perpendicular to d at its midpoint and extending horizontally in both directions. If the midline is taken as $0°$, at what other angle(s) θ is a maximum signal detected? A minimum signal? Assume all measurements are made much farther than 8.0 m from the antenna towers.
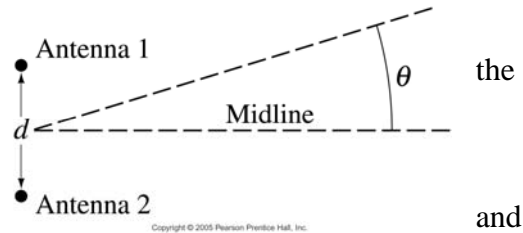
The wavelength of the signal is

$$\lambda + \frac{v}{f} = \frac{\left(3.00 \times 10^8 \ \mathrm{m/s}\right)}{\left(102.1 \times 10^6 \ \mathrm{Hz}\right)} = 2.94 \ \mathrm{m}.$$

Because measurements are made far from the antennae, we can use the analysis for the double slit.

For constructive interference, the path difference is a multiple of the wavelength:

$d \sin \theta = m \lambda, \, m = 0, 1, 2, 3, \ldots;$

$(8.0 \ \mathrm{m}) \sin \theta_{1 \, max} = (1)(2.94 \ \mathrm{m})$ which gives $\theta_{1 \, max} = 22°;$

$(8.0 \ \mathrm{m}) \sin \theta_{2 \, max} = (2)(2.94 \ \mathrm{m})$ which gives $\theta_{2 \, max} = 47°;$

$(8.0 \ \mathrm{m}) \sin \theta_{3 \, max} = (3)(2.94 \ \mathrm{m})$ which gives $\sin \theta_{3 \, max} > 1,$ so there is no third maximum.

Because the interference pattern will be symmetrical above and below the midline and on either side

of the antennae, the angles for maxima are $\boxed{22°, 47°, 133°, 158° \text{ above and below the midline.}}$

For destructive interference, the path difference is an odd multiple of half a wavelength:

$d \sin \theta = \left(m - \frac{1}{2}\right)\lambda, m = 1, 2, 3, \dots; \text{or}$

$(8.0 \,\text{m})\sin\theta_{1\,\text{min}} = \left(1 - \frac{1}{2}\right)(2.94 \,\text{m})$ which gives $\theta_{1\,\text{min}} = 11°$;

$(8.0 \,\text{m})\sin\theta_{2\,\text{min}} = \left(2 - \frac{1}{2}\right)(2.94 \,\text{m})$ which gives $\theta_{2\,\text{min}} = 33°$;   or

$(8.0 \,\text{m})\sin\theta_{3\,\text{min}} = \left(3 - \frac{1}{2}\right)(2.94 \,\text{m})$ which gives $\theta_{3\,\text{min}} = 67°$;

$(8.0 \,\text{m})\sin\theta_{4\,\text{min}} = \left(4 - \frac{1}{2}\right)(2.94 \,\text{m})$ which gives $\sin\theta_{4\,\text{min}} > 1$, so there is no fourth minimum.

Because the interference pattern will be symmetrical above and below the midline and on either

side of the antennae, the angles for minima are

$\boxed{11°, 33°, 67°, 113°, 147°, 169° \text{ above and below the midline.}}$

9. Light of wavelength 5.0×10^{-7} m passes through two parallel slits and falls on a screen 4.0 m away. Adjacent bright bands of the interference pattern are 2.0 cm apart.
   (a) Find the distance between the slits.
   (b) The same two slits are next illuminated by light of a different wavelength, and the fifth-order minimum for this light occurs at the same point on the screen as the fourth-order minimum for the previous light. What is the wavelength of the second source of light?

(*a*)   For constructive interference, the path difference is a multiple of the wavelength:

$d \sin \theta = m\lambda, m = 0, 1, 2, 3, \dots.$

We find the location on the screen from

$y = L \tan \theta.$

For small angles, we have

$\sin\theta \approx \tan\theta,$ which gives

$y = L\left(\dfrac{m\lambda}{d}\right) = \dfrac{mL\lambda}{d}.$

For adjacent bands, $\Delta m = 1$, so we have

$\Delta y = \dfrac{L\lambda\Delta m}{d};$

$2.0 \times 10^{-2} \,\text{m} = \dfrac{(4.0 \,\text{m})(5.0 \times 10^{-7} \,\text{m})(1)}{d}$, which gives $d = 1.0 \times 10^{-4} \,\text{m} = \boxed{0.10 \,\text{mm.}}$

(*b*)   For destructive interference, the path difference is given by

$d \sin \theta = \left(m + \frac{1}{2}\right)\lambda, m = 0, 1, 2, 3, \dots.$

Again we find the location on the screen from

$$y = L \tan \theta.$$

and again we use $\sin \theta \approx \tan \theta$, this time to obtain

$$y = \frac{\left(m + \frac{1}{2}\right) L \lambda}{d}.$$

We are told that

$$\frac{\left(5 + \frac{1}{2}\right) L \lambda_2}{d} = \frac{\left(4 + \frac{1}{2}\right) L \lambda_1}{d};$$

$$\left(5 + \frac{1}{2}\right)\lambda_2 = \left(4 + \frac{1}{2}\right)\left(5.0 \times 10^{-7}\,\text{m}\right)$$

which gives $\lambda_2 = \boxed{4.1 \times 10^{-7}\,\text{m}.}$

10. In a double-slit experiment, each slit has a width of 0.02 mm and they are spaced 0.14 mm apart. The pattern of light when a coherent 550 nm beam is incident on the slits is observed on a screen 4 m away.
   a. Find the spacing between interference fringes on the screen.
   b. Find the full width of the central diffraction maximum on the screen.
   c. How many fringes are visible within the central diffraction maximum?
   d. What happens to the pattern if the entire apparatus is immersed in water? Find new answers to each of the above parts.

   a. $\dfrac{dy_m}{D} = m\lambda \rightarrow \dfrac{\Delta y_m}{\Delta m} = \dfrac{\lambda D}{d} = \dfrac{550 \times 10^{-9}\,m \times 4m}{0.14 \times 10^{-3}\,m} = 15.7 \times 10^{-3}\,m = 15.7 mm$

   b. $\sin\theta = \dfrac{\lambda}{a} = \dfrac{y_D}{D} \rightarrow y_D = \dfrac{\lambda}{a} D = \dfrac{550 \times 10^{-9}\,m \times 4m}{0.02 \times 10^{-3}\,m} = 0.11m$ and the width is twice this
   distance, or 0.22m = 220mm.

   c. The number of fringes about the central maximum is given as the ration of 220mm to 15.7mm = 14. Including the central maximum gives us 15. The last two are actually at a diffraction minimum so are not visible. Thus the number of visible fringes is 15 – 2 = 13.

   d. Putting the apparatus under water changes only the wavelength of the light. The new wavelength is given as $\lambda_{water} = \dfrac{\lambda_{air}}{n_{water}} = \dfrac{550 \times 10^{-9}\,m}{1.33} = 4.13 \times 10^{-7}\,m = 413 nm$.

   Redoing parts a – c we find
   $$\dfrac{dy_m}{D} = m\lambda \rightarrow \dfrac{\Delta y_m}{\Delta m} = \dfrac{\lambda D}{d} = \dfrac{413 \times 10^{-9}\,m \times 4m}{0.14 \times 10^{-3}\,m} = 1.18 \times 10^{-3}\,m = 11.8 mm,$$

   $$\sin\theta = \dfrac{\lambda}{a} = \dfrac{y_D}{D} \rightarrow y_D = \dfrac{\lambda}{a} D = \dfrac{413 \times 10^{-9}\,m \times 4m}{0.02 \times 10^{-3}\,m} = 0.0826m = 82.6 mm \text{ (with a}$$
   width of 165.2mm) and the number of visible fringes is still 13 (165.2mm / 11.8mm = 14 fringes plus the central maximum for a total of 15.)

11. In a double-slit experiment with 500 nm light, the slits each have a width of 0.1 mm.
    a. If the interference fringes are 5 mm apart on a screen which is 4 m from the slits, determine the separation of the slits.
    b. What is the distance from the center of the pattern to the first diffraction minimum on one side of the pattern?
    c. How many interference fringes will be seen within the central maximum in the diffraction pattern? Draw a sketch of the pattern.

    a. $\dfrac{dy_m}{D} = m\lambda \to d = \dfrac{\Delta m\lambda D}{\Delta y_m} = \dfrac{500\times10^{-9}\,m\times 4m}{5\times10^{-3}\,m} = 0.4\times10^{-3}\,m = 0.4mm$

    b. $\sin\theta = \dfrac{\lambda}{a} = \dfrac{500\times10^{-9}\,m}{0.1\times10^{-3}\,m} = 0.005 \to y = D\sin\theta = 4m\times 0.005 = 0.02m = 20mm$

    c. $y_1 = \dfrac{D\lambda}{d} = \dfrac{500\times10^{-9}\,m\times 4m}{0.4\times10^{-3}\,m} = 5\times10^{-3}\,m = 5mm$. Thus in 20 mm there are bright fringes every 5mm, so we have a total of 4. This gives in the envelope, a total of 9 (the central maximum and 4 on each side of the central maximum.) However, the last two on each side of the central maximum are not visible. Therefore the number of visible spots is 7.

12. In a double slit experiment with two slits of width 1.0 μm spaced 4 μm apart, suppose a beam of electrons is incident on the slits after being accelerated from rest through a potential difference of 100 V.
    a. What is the wavelength of the electrons?
    b. If the pattern of detected electrons is observed on a fluorescent screen 20 m from the slits, what is the width of the central diffraction maximum?
    c. How many interference fringes will be observed within the central diffraction maximum?

    a. The wavelength of a particle is given by the DeBroglie relation. Thus we have $\lambda = \dfrac{h}{p} = \dfrac{6.63\times10^{-34}\,Js}{9.11\times10^{-31}\,kg\times 5.93\times10^{6}\,\frac{m}{s}} = 1.22\times10^{-10}\,m$ where the velocity is determined from the potential difference that the proton was accelerated through.

    Thus $\tfrac{1}{2}mv^2 = q\Delta V \to v = \sqrt{\dfrac{2q\Delta V}{m}} = \sqrt{\dfrac{2\times1.6\times10^{-19}\,C\times100V}{9.11\times10^{-31}\,kg}} = 5.93\times10^{6}\,\frac{m}{s}$.

    b. $\sin\theta = \dfrac{\lambda}{a} = \dfrac{1.22\times10^{-10}\,m}{1\times10^{-6}\,m} = 0.000122 \to y = D\sin\theta = 20m\times 0.000122 = 0.0024m = 2.4mm$ and the width of the central diffraction maximum is given as twice this distance, or 4.8mm.

    c. $y_1 = \dfrac{D\lambda}{d} = \dfrac{1.22\times10^{-10}\,m\times 20m}{4\times10^{-6}\,m} = 6.1\times10^{-4}\,m = 0.61mm$. Thus in 2.44 mm there are bright fringes every 0.61mm, so we have a total of 4. This gives in the envelope, a total of 9 (the central maximum and 4 on each side of the central maximum.) However, the last two on each side of the central maximum are not visible. Therefore the number of visible spots is 7.