The Solow Growth Model

**Version 1: No population growth, no technological progress**

Consider the following production function in which $Y$ is real GDP, $K$ is the capital stock, and $N$ is the labor force:

$$ Y = F(K, N). \quad (1.1) $$

Assume that the above is a constant-returns-to-scale production function. We can, therefore, write (1.1) as:

$$ \frac{Y}{N} = F(K/N, 1). \quad (1.2) $$

Let $y \equiv \frac{Y}{N}$, and $k \equiv \frac{K}{N}$. We can then re-write (1.2) as:

$$ y = f(k). \quad (1.3) $$

Assume that this is a closed economy with no government. Now, demand for goods and services in this economy can be written as:

$$ y = c + i. \quad (1.4) $$

Where $c$ and $i$ are consumption per worker and investment per worker, respectively. In the Solow model consumers save a fraction $s$ of their income. Therefore, $c = (1 - s)y$. Consequently, from (1.5), (1.4) can be written as

$$ y = (1 - s)y + i. \quad (1.6) $$

Now, by a simple re-arrangement of terms in (1.6), we get:

$$ i = sy. \quad (1.7) $$

Now, assume that the depreciation rate of capital is $\delta$. Keeping in mind that $sy$ in (1.7) can be
written equivalently as \( sf(k) \), the change in the capital stock (per worker) can then be written as:

\[
\Delta k = sf(k) - \delta k.
\]  

(1.8)

The steady-state value of \( k \), denoted by \( k^* \), is given by (1.8), when \( \Delta k \) is set to zero.

**The Golden Rule level of capital**

The steady-state value of \( k \) that maximizes \( c \) is called the *Golden Rule* level of capital.

Policymakers may be able to choose \( s \) such that, in steady state, \( c \) is maximized. Now, note that in steady state

\[
c^* = f(k^*) - \delta k^*.
\]  

(1.9)

A necessary condition for maximum \( c^* \) is

\[
f'(k^*) - \delta = 0
\]  

(1.10)

So, to find the *Golden Rule saving rate*, the following two equations, derived from (1.8) and (1.10), must be solved:

\[
sf(k^*) = \delta k \quad \text{and} \quad f'(k^*) = \delta.
\]  

(1.11)

Some highlights of the Solow model:

1. At the steady state, the (per capita GDP) growth rate is zero.
2. When \( k < k^* \) positive growth takes place (and vice versa).
3. An increase in \( s \) will lead to short-term positive growth. The steady-state growth will be zero but at a higher \( y \).

- **Version 2: Positive population growth, no technological progress**

Now, let population and labor force grow at a constant rate \( g_N \). Equation (1.8) must now be rewritten as
\[ \Delta k = sf(k) - (\delta + g_N)k. \]  

(2.1)

The steady-state condition is

\[ sf(k) - (\delta + g_N)k = 0. \]  

(2.2)

And the Golden Rule saving rate can be derived by solving (2.2) and

\[ \frac{dc^*}{dk} = d\left[ f(k*) - (\delta + g_N)k* \right]/dk = 0 \quad \Rightarrow \quad f'(k*) = (\delta + g_N). \]  

(2.3)

An additional insight in this version of the Solow model is that, all else equal, a country with a high rate of population growth will have a low steady state \( k \), and therefore a low level of \( y \).

- **Version 3: Positive population growth, positive technological progress**

Re-write the production function in (1.1) as

\[ Y = F(K, A \times N). \]  

(3.1)

Where \( A \) is the efficiency of labor, and the term \( A \times N \) is the labor force measured in efficiency units. Assume that technological progress causes \( A \) to grow at the rate of \( g_A \). In this model the number of efficiency units of labor is growing at the rate \( g_A + g_N \). Now, let

\[ y = \frac{Y}{(A \times N)} \] stand for output per efficiency unit of labor. The production function, as before, can be written as \( y = f(k) \). Equation (1.8), previously modified to (2.1), must now be written as

\[ \Delta k = sf(k) - (\delta + g_A + g_N)k. \]  

(3.2)

The steady-state condition is

\[ sf(k) - (\delta + g_A + g_N)k = 0. \]  

(3.3)

And the Golden Rule saving rate can be obtained by solving (3.3) and

\[ \frac{dc^*}{dk} = d\left[ f(k*) - (\delta + g_A + g_N)k* \right]/dk = 0 \quad \Rightarrow \quad f'(k*) = (\delta + g_A + g_N). \]  

(3.4)

Note that in this version of the Solow model \( y \) grows at the rate of \( g_A \) in the steady state.