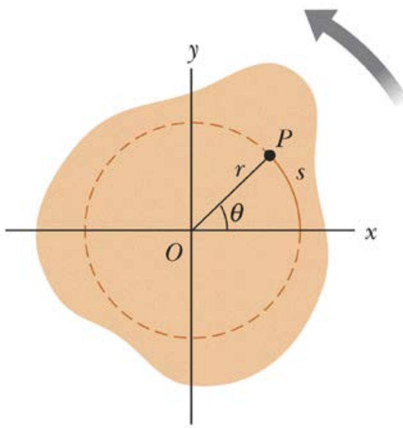
General Description of Motion

- We’ve seen that the translational motion of a complicated object can be accounted for by the motion of the center of mass
- Now, we turn to all the other motions with respect to coordinate system moving with the center of mass
- These are of two types: coherent and incoherent with coherent motions being rotations and vibrations that occur in a coordinated way, while incoherent motions are random thermal vibrations connected with the object’s internal energy or temperature
- If the object is rigid, then the overall motion consists of translation and rotation about the center of mass
- Because of this nice separation, we start our discussion of rotational motion by looking at a rigid object that rotates about a fixed axis of rotation

Rotation about a Fixed Axis

- Let's start with an object that is rotating about a fixed axis of rotation. How can we best describe it's motion?
- Rather than using {x,y,z}, it should make sense to use cylindrical coordinates {r,θ,z} or in 2-D just {r,θ}, since every point (P) in the object just travels around in a circle of radius r.
- So really, this is a problem with only one variable, the angle θ.
- Using the relation s = rθ for the arc length that P moves through to define the angle (in radians) $\theta = \frac{s}{r}$



Angular velocity

- Let's define the average angular speed of the object as

$$\bar{\omega} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

- Then we can define the instantaneous angular velocity in the usual way:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

- in units of rad/s

- can be + or –

- Note that there is a connection with the linear tangential speed given by the first derivative of the equation $s = r\theta$ so that $v = r\omega$ – but that the better velocity is ω since it is the same for the entire object

Angular acceleration

- Similarly, we introduce the average angular acceleration:

$$\bar{\alpha} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

- And the instantaneous angular acceleration:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

  - units are rad/s^2

- Again, the connection with linear variables comes from the second derivative of $s = r\theta$ so that $a_t = r\alpha$, where this linear acceleration is tangential - but that again the better variable to use is the angular acceleration, since it is a constant for the whole object
- Note that for this circular motion there is also a centripetal (or radial) acceleration given by $a_r = v^2/r = \omega^2 r$, so that the net acceleration is $\vec{a} = \vec{a}_t + \vec{a}_r$

Equations of Rotational Motion

- Using our definitions we can get equations of motion: (assuming α = constant)

$$\alpha = \frac{d\omega}{dt} \Rightarrow \int_{\omega_i}^{\omega_f} d\omega = \int_0^t \alpha dt \Rightarrow \boxed{\omega = \omega_i + \alpha t}$$

- And $$\omega = \frac{d\theta}{dt} \Rightarrow \int_{\theta_i}^{\theta_f} d\theta = \int_0^t \omega dt = \int_0^t (\omega_i + \alpha t) dt \quad or$$

$$\boxed{\theta = \theta_i + \omega_i t + \tfrac{1}{2}\alpha t^2}$$

- And eliminating t:

$$\boxed{\omega^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)}$$

Examples

TABLE 10.1 **A Comparison of Equations for Rotational and Translational Motion: Kinematic Equations**

Rotational Motion About a Fixed Axis with α = Constant (Variables: θ_f and ω_f)	Translational Motion with a = Constant (Variables: x_f and v_f)
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$
$\omega_f{}^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f{}^2 = v_i^2 + 2a(x_f - x_i)$

Example 7.1 A stationary exercise bicycle wheel starts from rest and accelerates at a rate of 2 rad/s^2 for 5 s, after which the speed is maintained for 60 s. Find the angular speed during the 60 s interval and the total number of revolutions the wheel turns in the first 65 s.

Rotational Kinetic Energy

- What's the KE for a particle traveling in a circle or radius r at constant speed?

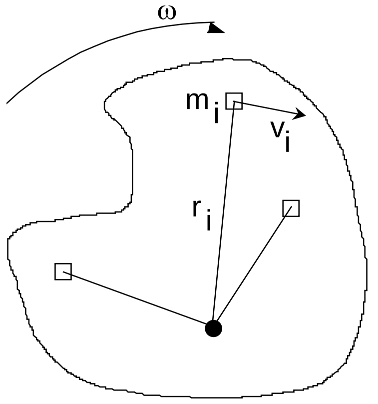
$KE_{rot} = ½\ mv^2 = ½\ m(r\omega)^2 = ½\ (mr^2)\omega^2 = ½\ I\omega^2$,

where $I = mr^2$ and ω = constant angular v

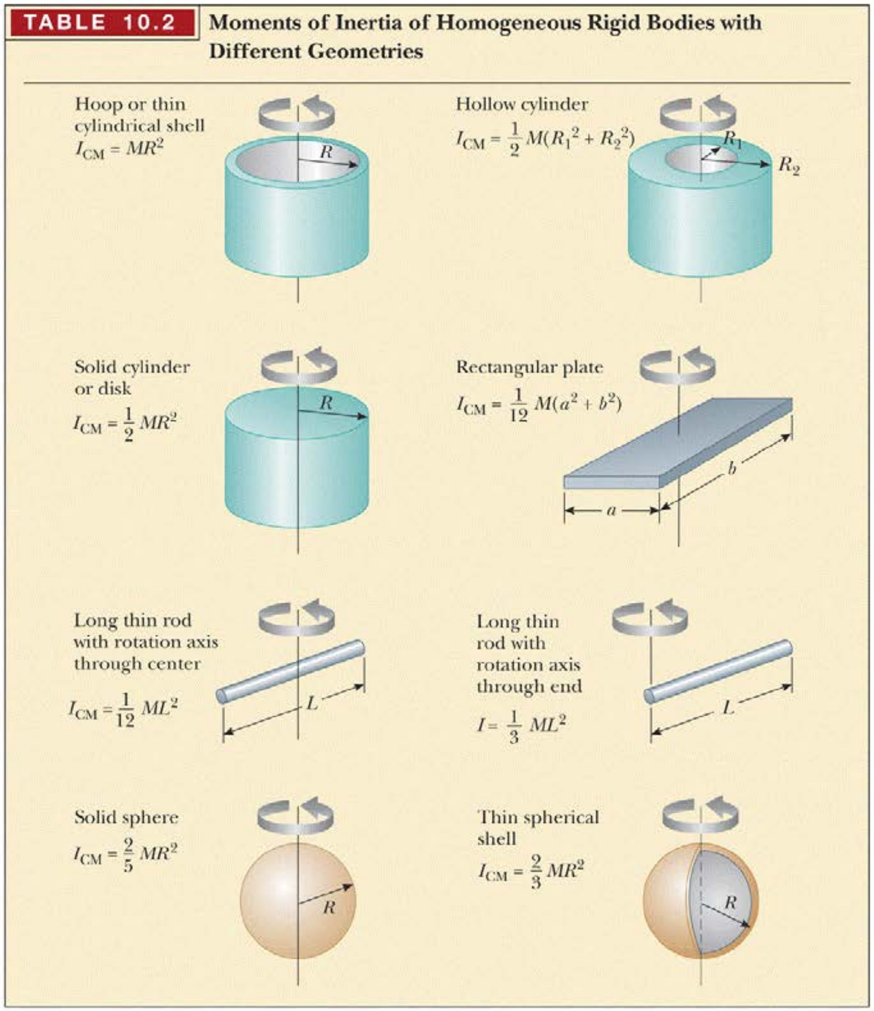
- I is called the moment of inertia and has units of kg-m^2 – it's a measure of the resistance to change in rotational speed and depends not only on the mass, but how it is distributed about the rotation axis
- With many particles we can repeat this derivation inserting a sum Σ to find that

$KE_{rot} = ½\ (\Sigma m_i r_i^2)\ \omega^2 = ½\ I\omega^2$ or with a continuous object:

$$I = \lim_{\Delta m \to 0} \Sigma r_i^2 \Delta m_i = \int r^2 dm = \int \rho r^2 dV$$



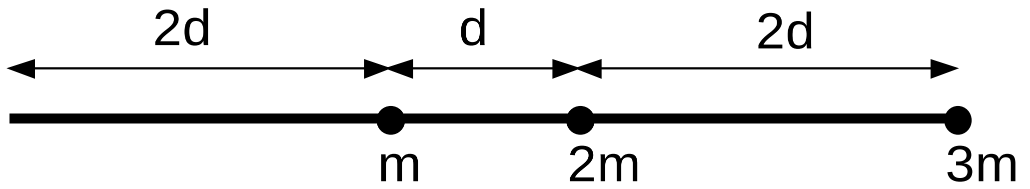
Moments of Inertia



All are of the form $I = kmd^2$ where d is the appropriate spatial dimension and k is some numerical coefficient

Example

- Example 7.4 Calculate the moment of inertia of the gadget shown. The small masses are attached by a light rigid rod and pivot about the left end of the rod. Use a value of $m = 1.5$ kg and $d = 0.2$ m.
- If the assembly were to pivot about its mid-point, find the moment of inertia about this axis as well.



Conservation of Energy

- When all the forces acting on a system of rigid bodies are conservative so that the work done by those forces can be expressed as a potential energy difference, we can write the conservation of energy equation for the system, composed of translating, rotating, or rolling symmetric rigid bodies, as

$$\tfrac{1}{2}m\mathrm{v}^2 + \tfrac{1}{2}I\omega^2 + PE = E = constant.$$

Example Problem

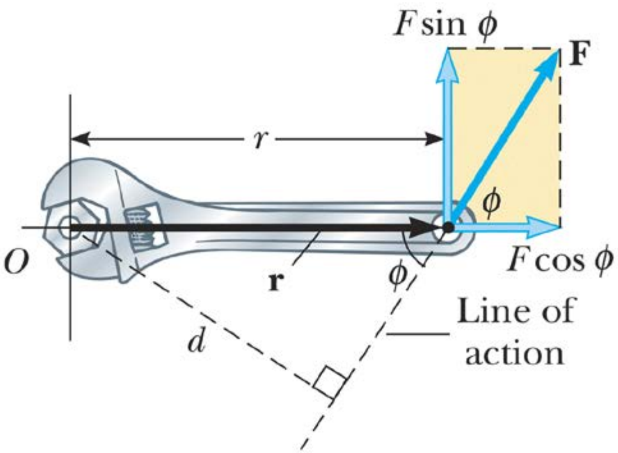
- Example 7.6 An empty bucket of 1 kg mass, attached by a light cord over the pulley for a water well, is released from rest at the top of the well. If the pulley assembly is a 15 cm uniform cylinder of 10 kg mass free to rotate without any friction, find the speed of the bucket as it hits the water 12 m below.

Torque

- How can we produce a rotation of a rigid object? Need a force, but it must be “well-placed” as well
- Torque, τ = rFsinϕ

  or $\tau = r_{\perp}F = rF_{\perp}$ (in N-m);

  $r_{\perp}$ = moment arm
- Torque is the rotational analog of force – it depends on F but also on where the force is applied



Rotational Work and Energy

- Starting from the work-energy theorem

W_{net} = ΔKE, if the object can only rotate about a fixed axis we have, in a time interval Δt,

$$\Delta W_{net,external} = \Delta KE_{rot} = \Delta \tfrac{1}{2} I \omega^2$$

So that

$$\Delta W_{net,external} = \tfrac{1}{2} I (\omega + \Delta\omega)^2 - \tfrac{1}{2} I \omega^2$$

Taking the difference and only keeping terms in $\Delta\omega$, we get

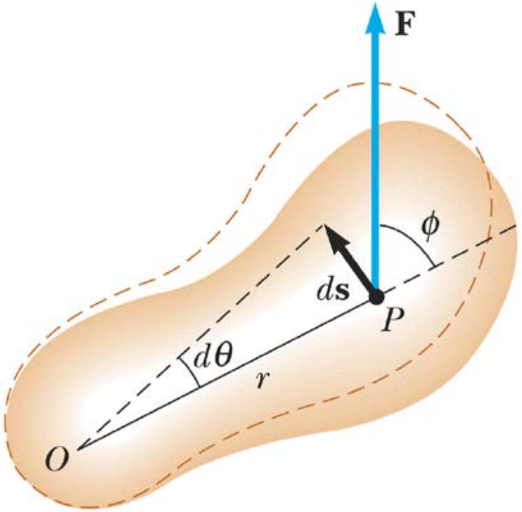
$$\Delta W_{net,external} = I \omega \Delta\omega$$

or

$$\Delta W_{net,external} = I \omega \frac{\Delta\omega}{\Delta t} \Delta t = \left(I \alpha \right) \Delta\theta$$

Rotational Work and Energy II

- Now, from the general definition of work $$\Delta W_{net,ext} = \left(F_{net,ext}\right)_x \Delta x$$



- But for a particle traveling in a circle, the displacement is

$$\Delta x = s = r\Delta\theta$$

so that

$$\Delta W_{net,ext} = \left(F_{net,ext}\right)_\perp r\Delta\theta = \tau_{net,ext}\Delta\theta$$

- Looking back we see that $$\tau_{net,ext} = I\alpha$$

in analogy with $F_{net,\ ext} = m\mathit{a}$

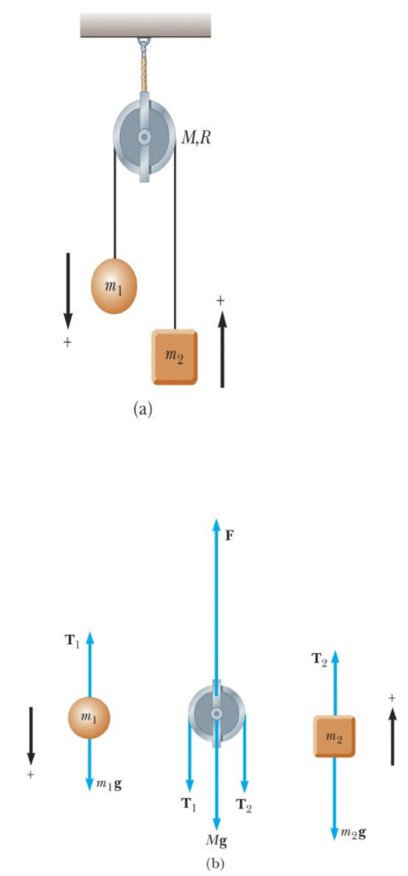
Quick Review

- We introduced the rotational variables: θ, ω, α and found analogous equations relating them to those from linear variables
- We next introduced the moment of inertia of a particle $I = mr^2$ and its generalizations and found the rotational $KE_{rot} = ½\, I\omega^2$
- We then introduced the analog of F, namely the torque $\tau = rF\sin\phi = r_\perp F = rF_\perp$
- We saw that $\tau = I\alpha$, the analog to F = ma
- Also Work = $\tau\,\Delta\theta$ in analogy with Fx

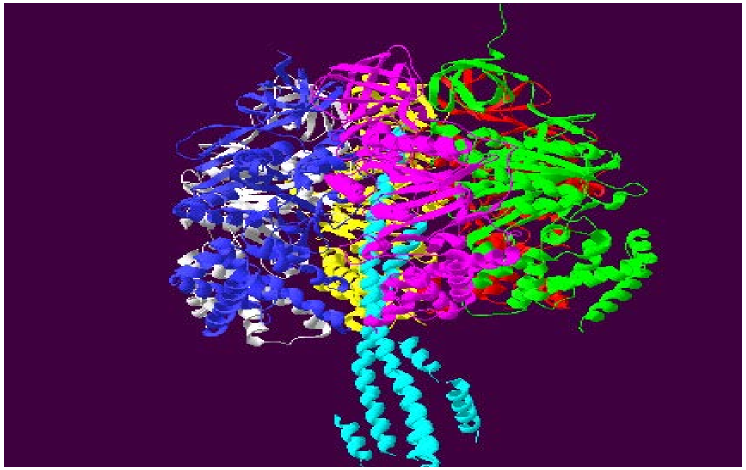
Example Problems

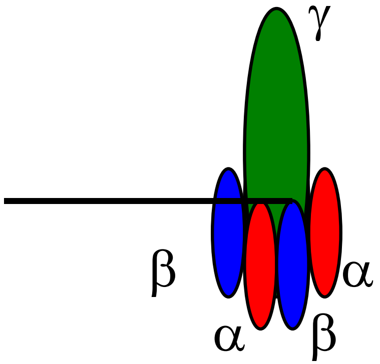
- Atwood machine with real pulley – Find the acceleration of the masses

- Race between a hoop and cylinder of the same mass and radius down an inclined plane from a height H without slipping – which one wins?



Bio-example: F1-ATPase





• lower the ATP concentration - individual step rotations of 120^o of the shaft were observed.
• torque measured for each step rotation was 44 pN-nm,
• calculated the work done by this rotary motor in each step rotation: a step rotation angle of $\Delta\theta$ = 120^o = ($2\pi/3$)
• they found that ΔW = ($2\pi/3$)(44 pN-nm) = 92 pN-nm = 92 x 10^{-21} J.
• This value is very close to the energy liberated by one ATP molecule when it is hydrolyzed to ADP.
• smallest of all rotary motors is nearly 100% efficient in converting energy into rotational work

Rotational analogs

- Remember that we were able to re-write Newton's second law **F** = m**a** in terms of the momentum as **F** = d**p**/dt. Here, **F** is the net external force on the system and **p** is the total momentum of the system. This was particularly useful when $\mathbf{F}_{\text{net, ext}} = 0$, so that momentum was conserved.
- Now that we have a rotational form of the equation $\tau = I\alpha$, what is the corresponding "momentum" equation? And does it lead to a new conservation law?

Angular Momentum for a particle

- By analogy, you might guess correctly that angular momentum L is given by L = $I\omega$.
- For a particle $I = mr^2$, so that we have L = $mr^2\omega$ = $rm(r\omega)$ = r(mv) = rp
- We can write that $\boldsymbol{\tau} = d\mathbf{L}/dt$, the analog to $\mathbf{F} = d\mathbf{p}/dt$
- This can also be written as $\boldsymbol{\tau} = d\mathbf{L}/dt = I d\omega/dt = I\alpha$

L for a system

- For a system, the generalization is

  $\boldsymbol{\tau}_{\text{net, external}} = d\mathbf{L}_{\text{total}}/dt$, with the internal torques canceling pair-wise, just as the internal forces do. Here, $\mathbf{L}_{\text{total}} = \sum \mathbf{L}_i$.

- For a rigid body rotating about a fixed axis, all points move in circles and r and v are perpendicular to each other, so

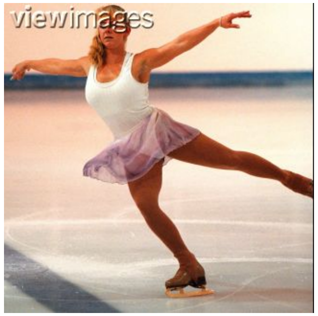
  $|\mathbf{L}_{\text{total}}| = \sum m_i v_i r_i = \sum m_i(\omega_i r_i) r_i = \sum (m_i r_i^2)\omega_i$

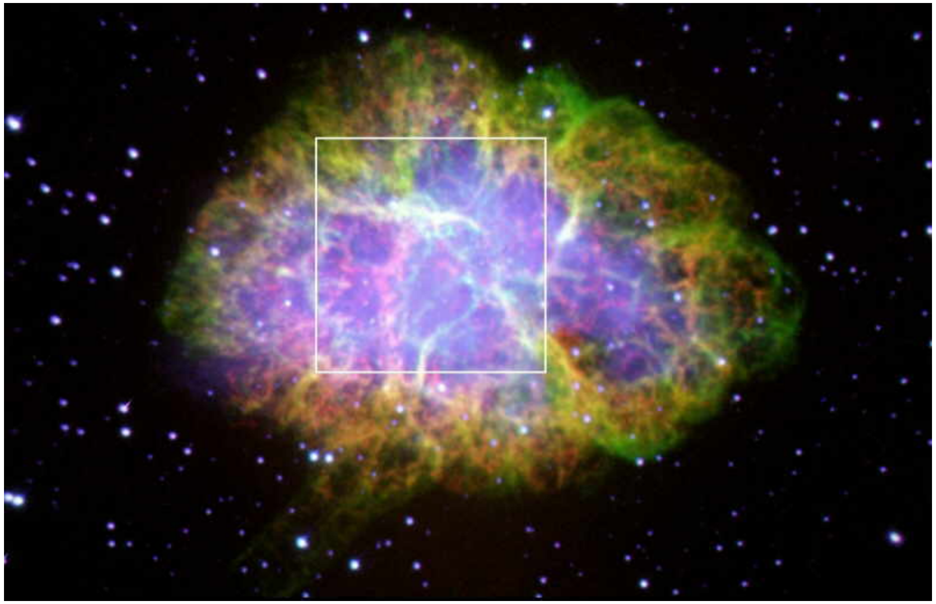
  $= I_{\text{total}}\ \omega$

- This is the rotational analog of p = mv

Conservation of Angular Momentum

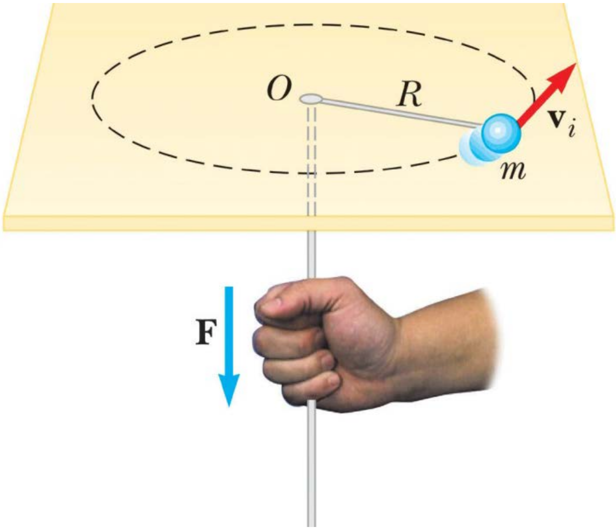
- Since $\boldsymbol{\tau}_{\text{net, external}} = d\mathbf{L}/dt$, if $\tau_{\text{net, external}} = 0$ or the system is isolated, then $\mathbf{L}$ = constant, and we have a new conservation law.
- Examples:
  - Isolated platform
  - Ice skater – no friction
  - Diving off diving board
  - Neutron stars





Problems

Ball of mass m on a frictionless table. Initially set in circular motion at radius R and speed v_i. Then we pull the string in so the radius shrinks to r. What is the final speed of the ball and is KE conserved?



Merry-go-round problem (Ex. 7.14):

A 5 m radius merry-go-round with frictionless bearings and a moment of inertia of 2500 kg-m^2 is turning at 2 rpm when the motor is turned off. If there were 10 children of 30 kg average mass initially out at the edge of the carousel and they all move into the center and huddle 1 m from the axis of rotation find the angular speed of the carousel.

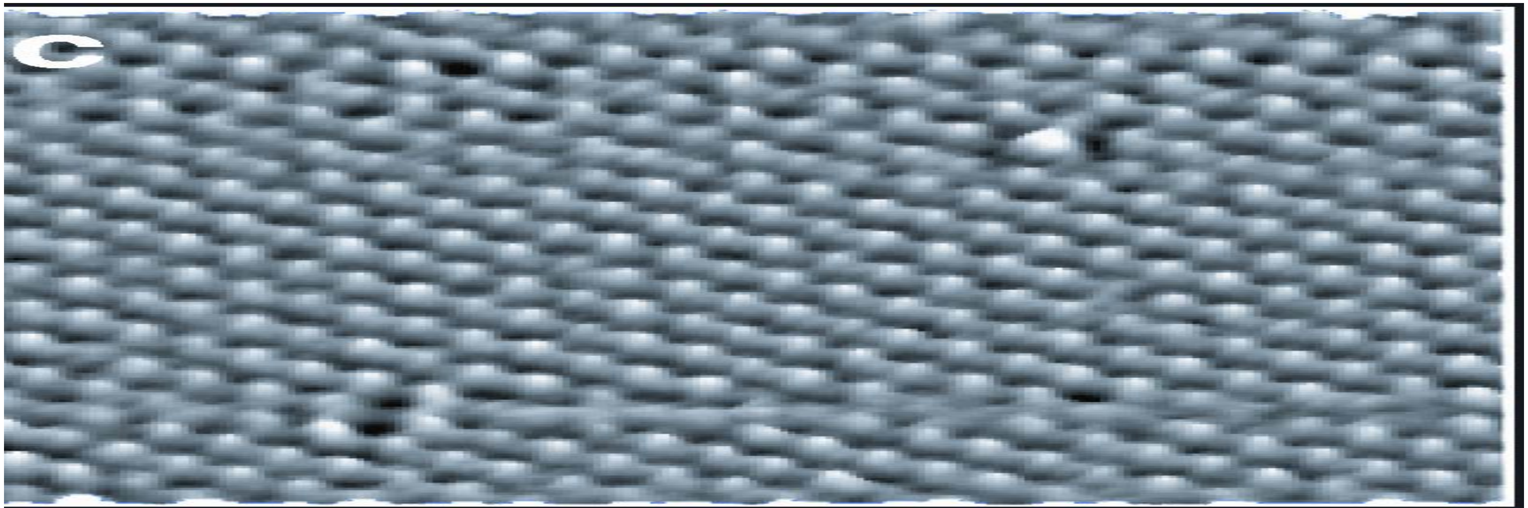
Summary

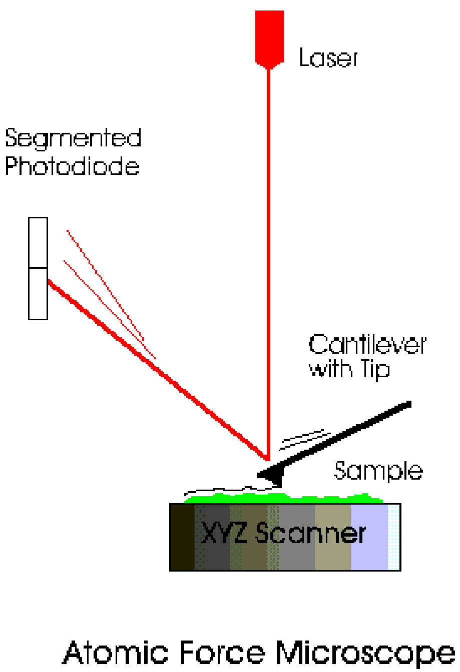
TABLE 10.3 **A Comparison of Equations for Rotational and Translational Motion: Dynamic Equations**[a]

	Rotational Motion About a Fixed Axis	Translational Motion
Kinetic energy	$K_R = \frac{1}{2}I\omega^2$	$K = \frac{1}{2}mv^2$
Equilibrium	$\sum \tau = 0$	$\sum \mathbf{F} = 0$
Newton's second law	$\sum \tau = I\alpha$	$\sum \mathbf{F} = m\mathbf{a}$
Newton's second law	$\sum \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$	$\sum \mathbf{F} = \frac{d\mathbf{p}}{dt}$
Momentum	$L = I\omega$	$\mathbf{p} = m\mathbf{v}$
Conservation principle	$L_i = L_f$	$\mathbf{p}_i = \mathbf{p}_f$
Power	$\mathcal{P} = \tau\omega$	$\mathcal{P} = Fv$

[a] Equations in translation motion expressed in terms of vectors have rotational analogs in terms of vectors. Because the full vector treatment of rotation is beyond the scope of this book, however, some rotational equations are given in nonvector form.

Atomic Force Microscopy





How is a macroscopic tip able to measure the surface height with sub-atomic resolution?

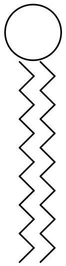
Effective spring constant for the cantilever is much smaller than the effective spring constant that holds the surface atoms together and the tip applies a very small (10^{-7} to 10^{-11} N) force on the surface Cantilevers used in AFM are usually microfabricated silicon made with integrated tips or with glued diamond tips with effective spring constants of 0.1 - 1.0 N/m.

Rotational Diffusion and Membranes

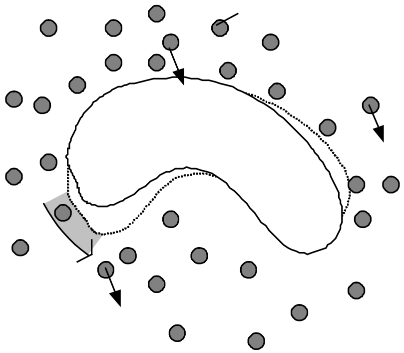
$$\tau_f = -f_R \omega.$$

$$f_R = 8\pi\eta r^3,$$

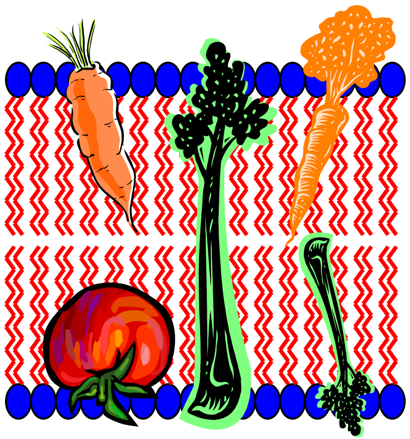
Rotating molecule feels frictional torque from surrounding viscous fluid



Fluid mosaic model of membrane



Cartoon of a macromolecule undergoing rotational diffusion due to random collisions with solvent molecules



Vegetable model of membrane

$$D_R = \frac{k_B T}{f_R},$$

Rotational diffusion coefficient

Static Equilibrium

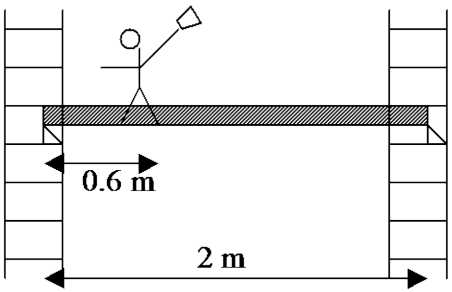
- $\vec{p}$ = *constant;* L = *constant* at equilibrium
- special cases when the object is at rest

$$F_{x,net} = 0 \qquad \tau_{net} = 0$$

$$F_{y,net} = 0.$$

- Problems:

1. (P7.33)A housepainter who weighs 750 N stands 0.6 m from one end of a 2.0 m long plank that is supported at each end by ladder anchors. If the plank weighs 100 N, what force is exerted upon each anchor?



Problem 2

- Suppose that a 50 N uniform crate at rest is pushed with a horizontal force of 30 N applied at the top of the crate with dimensions as shown below. If the coefficient of static friction is 0.7, will the crate slide along the surface or pivot at point O? If it will pivot, find the minimum applied force that will make the crate pivot about O.