PHYSICS 122
Winter 2012

Uncertainty Calculations

This handout will remind you of a number of basic ideas on uncertainty analysis (and possibly introduce some new ones). Uncertainty is used here to refer to those sources of error in a measured quantity (or one derived from measured quantities) arising from random sampling errors rather than from any systematic error in measurement. These latter errors can be corrected for if they are known. For example, if it is discovered that a meter used to measure electric current is mis-calibrated and that it consistently reads 1.5% too small a value, then all of the current readings can be systematically increased by this correction factor.

The discussion below has three parts: 1) how to estimate/calculate the uncertainty in a measured quantity; 2) how to set the number of significant figures to give in the result; and 3) how to propagate these uncertainties in order to find the uncertainty in a derived quantity.

Before beginning, two issues should be mentioned.

First, the words ***uncertainty***, or ***inexactness***, or ***error***, do not refer to mistakes made by the experimenter, but rather to the inherent randomness found in nature, a fundamental part of the measurement process. Repeated careful measurements of the same quantity will undoubtedly give a set of different results. Uncertainty refers to the range of possible results for the measured quantity.

Second, in thinking about this you should make a clear distinction in your mind between the notion of ***precision*** and ***accuracy***. Precision refers to the number of justifiable significant figures in your results, and is a measure of the care and the degree of sophistication used in the measurement, while accuracy refers to the how close your result is to the correct value. Unfortunately, it may happen that the precision of a measurement is higher than its accuracy, meaning that the estimate of the uncertainty in an answer may be too small and that the correct value differs from the result by substantially more than this estimated uncertainty. Often this arises when there are *systematic* errors that have not been accounted for in the experiment and these skew the results. If the measured quantity is not known then these systematic errors will only become apparent with further attempts to refine the measurement.

1. Uncertainties in a Measured Quantity

The best way to estimate the uncertainty in a quantity is to make repeated unbiased measurements using the highest precision (most careful and detailed) possible. Try to interpolate between the smallest units on the measuring device – e.g., read the meter stick to tenths of millimeters, for example – and repeat your measurements a good number of times. The uncertainty is an expression of the fluctuation in the measured set of numbers and is usually expressed as the ***standard deviation***, or root mean square (***RMS***) ***deviation***.

To find the RMS deviation in a quantity x, σ_x, you

Trial	Time
1	1.63
2	1.62
3	1.65
4	1.69
5	1.63
6	1.65

1. find the average $< x >= \frac{1}{N}\sum_{i=1}^{N} x_i$;
2. calculate the deviation of each data point from the average $d_i =< x > - x_i$;
3. square each deviation;
4. calculate the average of the squares of the deviations; and
5. take the square root so that $\sigma_x = \sqrt{\left(\frac{1}{N}\right)\sum_{i=1}^{N}\left(< x > - x_i\right)^2}$.

Trial	Dev	Dev^2
1	-0.015000	0.000225
2	-0.025000	0.000625
3	0.005000	0.000025
4	0.045000	0.002025
5	-0.015000	0.000225
6	0.005000	0.000025

Note that the RMS deviation follows the reverse order in the expression "RMS deviation," finding first the squares of the deviations from the mean, then their mean, and finally the square root. In the example shown in the tables, the average value of the time is 1.645 sec, and the RMS deviation is 0.02291 sec. You might want to pause at this point and confirm these values.

As a result, the time could be stated as 1.645 ± .0229 sec. The uncertainty is the expression of the range of values of time, such that about 2/3 of the data points fall with the stated range. A histogram of the repeated measurements, showing the number of data points with a given time result vs. the time will yield, if there are a large enough set of data points, a distribution that has a smooth "**bell shape**," or **gaussian shape**, centered at the mean value. Only if enough data points have been recorded, and if they are truly random fluctuations will this be true. The standard deviation is a measure of the width of this curve from <x> - σ_x to <x> + σ_x, within which 2/3 of the data will be found. This also means that, on the average, there will be about 1/3 of the data points that will fall outside the stated range. Its significance is that it predicts that if a single additional measurement of x is made then there is a 67% probability that this single result will be within 1 standard deviation of the measured average value.

Note that the standard deviation does not get smaller as the number of measurements increases since the width of the gaussian distribution does not change. In this sense it does not tell you how well you know the average value of x itself, but rather how well the shape of the gaussian is known. A second statistical quantity, the ***standard deviation of the mean***, tells us how well the average is known. This quantity is defined as $\sigma_{<x>} = \frac{\sigma_x}{\sqrt{N}}$ and does decrease as N increases. Thus, an average value should be reported as (<x> ± σ_x) with the number of measurements specified, or, better, as (<x> ± $\sigma_{<x>}$), indicating which is given.

There are two caveats to the above discussion. First, it turns out the in calculating the standard deviation, a better estimate is given by dividing by (N-1) in place of (N) so that the standard deviation is given by $\sigma_x = \sqrt{\left(\frac{1}{N-1}\right)\sum_{i=1}^{N}\left(< x > - x_i\right)^2}$. Second, if you can't make at least a small number of repeated measurements then you will have to make your own educated guess for the uncertainty in a measured value.

2. Significant Figures

The usual rule is to ***keep only one significant figure in the uncertainty value*** and to ***round off the measured value appropriately, keeping digits out to the same decimal place as the uncertainty***. A few examples should clarify this, but first we mention that this procedure should only be used when reporting final values. In other words, you should keep more digits than the significant digits in your calculations (calculators automatically do this for you so long as you do a continuous calculation without re-entering numbers with fewer digits.) This is done so as to avoid round-off errors until the final desired number is obtained. Only then should the number be rounded off according to the uncertainty.

Examples:

If your calculator gives you the following average values and uncertainty values (standard deviations of the mean) – in the left column, then you should report these results as shown in the right column:

14.534632 ± 0.6854 cm	14.5 ± 0.7 cm
965.4372 ± 1.483 cm/s^2	965 ± 1 cm/s^2
$2.67348 \times 10^6 \pm 8.362 \times 10^4$ m	$(2.67 \pm 0.08) \times 10^6$ m
$1.5962 \times 10^{-4} \pm 4.5634 \times 10^{-6}$ A	$(1.60 \pm 0.05) \times 10^{-4}$ A

3. Uncertainty Propagation

Uncertainties can be expressed either as absolute, numerical values as above, or as fractional (or percent) uncertainties. ***In presenting results, usually absolute uncertainties are given*** as above, but ***in discussions of uncertainties, precision, and accuracy it is usually best to write about percent uncertainties*** since they are more telling. For example, an uncertainty of ±1 when the result is 10 represents a 10% uncertainty while when the result is 1000, it represents only a 0.1% uncertainty. The rules presented in this section show how to combine, or propagate, uncertainties from measured quantities so as to obtain the uncertainty in a derived quantity. These rules all come from calculus, as should be apparent, although we will not derive them here.

Rule 1: When you are calculating the product or quotient of two measured quantities (u = xy or u = x/y), and each has an uncertainty (σ_x and σ_y), then the fractional uncertainty in the product or quotient is the square root of the sum of the fractional (or %) errors:

$$\frac{\sigma_u}{u} = \sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$$

Example: Suppose we are calculating the area of a rectangle for which the length and width are

L = (10.2 ± 0.3) cm

W = (15.6 ± 0.1) cm.

The area is 159.12 cm^2, and the uncertainty in the area is

$$\sigma_{area} = \sqrt{(2.94\%)^2 + (0.64\%)^2} = 3.01\% \text{ or } 4.9 \text{ cm}^2$$

We might then give the area as

area = (159 ± 5) cm^2.

Rule 2: When you are calculating the sum of or difference between two measured quantities (x and y), and each has an uncertainty then the uncertainty σ_u, in the sum or difference (u = x+y or u = x-y), is the square root of the sum of the squares of the numerical uncertainties, i.e.

$$\sigma_u = \sqrt{\sigma_x^2 + \sigma_y^2}$$

Example: Calculate the perimeter of the rectangle given in the example above.
First we write the formula:

P = 2L + 2W

Always work from the inside to the outside. That is, first we double the length and width using rule (1). Note that there is no uncertainty in the factor of two, so the fractional uncertainty in 2L is the same as the fractional uncertainty in L . Thus

2L = 20.4 ± 2.9% or ± 0.6 cm

2W = 31.3 ± 0.64% or ± 0.2 cm.

Then we calculate P and it's uncertainty:

P = (51.6 ± 0.6) cm

Note that rules 1 and 2 can be combined and generalized.

For example, if f = x + yz/t, then (you should work through this calculation)

$$\sigma_f^2 = \sigma_x^2 + \left[\left(\frac{\sigma_y}{y} \right)^2 + \left(\frac{\sigma_z}{z} \right)^2 + \left(\frac{\sigma_t}{t} \right)^2 \right] \left(\frac{yz}{t} \right)^2 .$$

Rule 3: For more complicated functions, the rules of calculus tell you what to do. A few examples are:

if $y = x^n$ then $\frac{\sigma_y}{y} = n \frac{\sigma_x}{x}$,

and

if y = sin(θ) then $\sigma_y = \cos(\theta)\sigma_\theta$.

Without using calculus, an alternative method is to make numerical calculations in the following spirit. For example if y = A sin(θ), then we can write that

$y_1 = A \sin(\theta + \sigma_\theta)$ and $y_2 = A \sin(\theta - \sigma_\theta)$, and we can use $(y_1 - y_2)/2$ as a measure of the uncertainty in y. This method (called ***the max – min method***) usually works reasonably well because most functions are linear over small intervals.