Appendices*

for

R&D Policies, Endogenous Growth and Scale Effects

by

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Appendix A : Proof of Stability for Competitive Equilibrium

I characterize the transitional dynamics of the economy by generating an autonomous system for the equations of motion for c and d, where c is consumption per capita and $d \equiv D/N$ is R&D difficulty per capita. During the transition, the labor market conditions, free-entry in R&D and optimal rent protection activity conditions must hold. I restrict attention to the domain $d > 0$, and $\lambda(1 - s) > c > 0$ which ensures $\iota > 0$. Note from equation (13) that $\dot{\upsilon}/\upsilon = \dot{D}/D$. Substituting for $X(\omega)$ from (21) into (7) using $\iota = \iota(\omega)$ and $X = X(\omega)$ and the measure one of industries, it follows that $\dot{D}/D = [\delta s / \gamma d] + \iota\mu$. Combining (19) and (21), $w\gamma X/N$ can be written as $w\gamma X/N = \delta\iota a_\iota(1 - \phi_\iota)s/(1 + \mu\iota)\gamma$. Solving equation (11) for r and substituting the resulting expression into (4), using the expressions for $\dot{\upsilon}/\upsilon$ and $w\gamma X/N$ immediately implies:

$$\dot{c} = c\left\{ \frac{1}{d}\left[\frac{c(\lambda-1)}{\lambda a_\iota(1-\phi_\iota)} - \frac{\delta s}{\gamma\left[\mu + (1/\iota(c,d))\right]} + \frac{s\delta}{\gamma} \right] - \iota(c,d)\left[1-\mu\right] - \rho \right\}$$

where $\iota(c, d) = [1 - s - (c/\lambda)]/da_\iota$ comes from (20). Observe that $dd/dc|_{\dot{c}=0}$ is indeterminate. On the other hand, it is possible to draw some inferences on the movement of c around $\dot{c} = 0$. More specifically, if $\mu < 1$ then $d\dot{c}/dc > 0$, implying that starting from any point on $\dot{c} = 0$ curve, an increase in c leads to $\dot{c} < 0$. If $\mu > 1$ then $d\dot{c}/dc$ remains ambiguous.

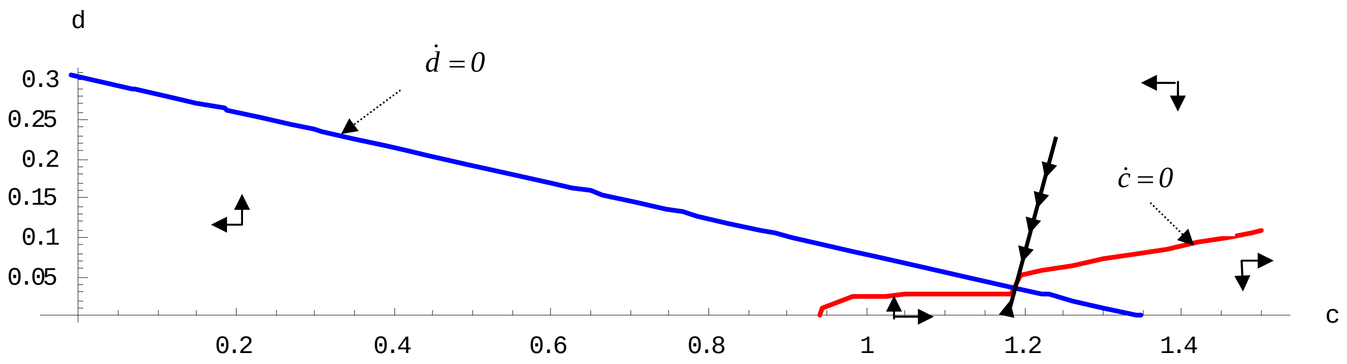
To find the equation of motion for d simply substitute $\iota(c, d)$ into $\dot{d}/d = \dot{D}/D - n$ using the expression for $\dot{D}/D$ from above. This implies:

$$\dot{d} = \frac{\delta s}{\gamma} + \frac{\mu}{a_\iota}\left[1 - s - \frac{c}{\lambda}\right] - nd$$

Observe that $dd/dc|_{\dot{d}=0} < 0$ and that $d\dot{d}/dd < 0$. Starting from any point on the $\dot{d} = 0$ line, an increase in d renders $\dot{d} < 0$, and an decrease in d renders $\dot{d} > 0$.

Given the ambiguous relationships coming from $\dot{c} = 0$, there are multiple possibilities. However, we know that in the relevant domain, the intersection of $\dot{c} = 0$ and $\dot{d} = 0$ generates a unique point. Hence, using a graphical approach, it is straightforward to analyze the local stability of this non-linear system. When $\dot{c} = 0$ is upward sloping, the system has a stable node if $d\dot{c}/dc > 0$ and a stable focus otherwise. When $\dot{c} = 0$ is downward sloping, the system has a stable node independent of the sign of $d\dot{c}/dc$. This result holds regardless of $\dot{c} = 0$ being flatter or steeper than $\dot{d} = 0$. Therefore, we conclude that the system is locally stable, exhibiting either a stable node or a stable focus. Figure

Figure Appendix A: Plot of $\dot{c} = 0$ and $\dot{d} = 0$



Appendix A plots the demarcation curves $\dot{c} = 0$ and $\dot{d} = 0$ under the benchmark values for the parameters. Observe that $\dot{c} = 0$ is upward sloping and given $\mu < 1$ it follows that $d\dot{c}/dc > 0$. In this case, the system is saddle path stable. Numerical simulations imply that for a wide range of reasonable parameters the picture depicted in Figure Appendix A remains valid.

Appendix B: Optimal R&D Policy: Solution, Uniqueness and Stability

The social planner's problem is to

$$\max_{\{\iota\}} \int_0^{\infty} e^{-(\rho-n)t} \{\Phi(t) \log \lambda + \log[1 - s - a_\iota d(t)\iota(t)]\} dt \qquad \text{(B1)}$$

subject to the state equation $\dot{\Phi} = \iota(t)$, $\dot{d} = (\delta s/\gamma) + \mu\iota(t)d(t) - nd(t)$, the initial conditions $\Phi(0) = 0$, $d(0) = d_0 > 0$, and the control constraint, $(1 - s)/a_\iota d(t) \geq \iota(t) \geq 0$ for all t. The current value Hamiltonian for this problem can stated as:

$$H = \Phi \log\lambda + \log[1 - s - a_\iota d\iota] + \theta\iota + \eta[(\delta s/\gamma) + \mu\iota d - nd].$$

For an interior solution, the first order condition is:

$$\frac{\partial H}{\partial \iota} = -\frac{a_\iota d}{1 - s - a_\iota d\iota} + \theta + \eta\mu d = 0 \,. \qquad \text{(B2)}$$

The costate equations are

$$\dot{\theta} = (\rho - n)\theta - \frac{\partial H}{\partial \Phi} = (\rho - n)\theta - \log \lambda \quad , \qquad \text{(B3)}$$

$$\dot{\eta} = (\rho - n)\eta - \frac{\partial H}{\partial d} = (\rho - n)\eta + \frac{a_\iota \iota}{1 - s - a_\iota d\iota} - \eta\left(\mu\iota - n\right). \qquad \text{(B4)}$$

I solve the optimal control problem for a Balanced-Growth Path (BGP) in which all endogenous variables grow at constant rates (although not necessarily the same rate) and the rate of innovation is positive $\iota > 0$. It follows from (7) that for $\dot{D}/D$ to be constant, X/D and ι must be constant as well. Given $\dot{X}/X = n$ by (21), for X/D to be constant, $\dot{D}/D = n$ must hold. This in turn implies that $d = D/N$ is constant. Substituting $R = \iota D$ into (20) implies that c must be a constant. Let $g_\theta = \dot{\theta}/\theta$. Constancy of g_θ requires θ be a constant, which in turn implies $\dot{\theta}/\theta = 0$. Let $g_\eta = \dot{\eta}/\eta$. With ι and d constant, constancy of g_η requires η be a constant, which in turn implies $\dot{\eta}/\eta = 0$.

Imposing $\dot{\theta} = 0$ and $\dot{\eta} = 0$ on (B3) and (B4) implies $\theta = \log\lambda/(\rho - n)$ and $\eta = a_\iota\iota/[1 - s - a_\iota d\iota)(\iota\mu - \rho)]$. Imposing $\dot{d} = 0$ yields $d = \delta s/[\gamma(n - \iota\mu]$. Note that (20) collapses to $c/\lambda = 1 - s - a_\iota d\iota$. Substituting for θ, η and d into (B2) using $c/\lambda = 1 - s - a_\iota d\iota$ gives (31). The RD^{SO} and LM conditions, given by (31) and (27) respectively, determine the optimal balanced growth levels $\tilde{\iota}$ and $\tilde{c}$. To see uniqueness, consider plotting the RD^{SO} and LM curves in (ι, c) space restricting the domain to $\lambda(1-s) > c > 0$ and $n/\mu > \iota > 0$. For the RD^{SO} equation: $(dc/d\iota)\big/_{RD^{SO}} > 0$. Moreover, as $\iota \to 0$, $c \to (c_0)^{SO} = \lambda s a_\iota(\rho - n)\delta/(\log\lambda)n\gamma$, and as $\iota \to \iota^{max} = n/\mu$, $c \to \infty$. For the LM equation: $(dc/d\iota)\big/_{LM} > 0$. Furthermore, as $\iota \to 0$, $c \to \lambda(1-s)$, and as $\iota \to \iota^{max} = n/\mu$, $c \to -\infty$. Hence, for a unique equilibrium, the intercept of the LM curve must be strictly higher than that of the RD^{CE} curve: $\lambda(1-s) > (c_0)^{SO} \Rightarrow \log\lambda > [s a_\iota(\rho - n)\delta] / [(1-s)n\gamma]$. Observe that this is quite similar to the uniqueness condition for competitive equilibrium which was $\lambda(1-s) > c_0 \Rightarrow \lambda - 1 > [s a_\iota(1-\phi_\iota)(\rho - n)\delta]/[(1-s)n\gamma]$.

To analyze stability, I invoke the transversality conditions:

$$\lim_{t\to\infty} e^{-(\rho-n)t}\theta(t)\Phi(t) = 0.$$

$$\lim_{t\to\infty} e^{-(\rho-n)t}\eta(t)d(t) = 0.$$

The transversality condition for θ and the costate equation (B3) imply that $\theta(t) = \log\lambda(\rho - n)$ for all $t > 0$ [see Grossman and Helpman (1991), p. 71 and 103]. To see this formally, note that the solution for $\dot\theta = (\rho - n)\theta - \log\lambda$ is $\theta(t) = \theta_0 e^{(\rho-n)t} - [\log\lambda\,(e^{(\rho-n)t} - 1)]/(\rho - n)$, where θ_0 is the initial value for θ, which I assume exists but is unknown. Substituting this into the transversality condition implies $\lim_{t\to\infty} e^{-(\rho-n)t}\theta(t)\Phi(t) = \{\theta_0 - [\log\lambda/(\rho - n)]\}\,\Phi(t) = 0$. Given $\lim_{t\to\infty}\Phi(t) = \infty$, the only possible solution to this limit is $\theta_0 = \log\lambda/(\rho - n)$. The solution to $\dot\theta$ then implies

$$\theta(t) = \frac{\log\lambda}{\rho - n}.$$

Substituting this into the transversality condition and using the L'Hopital's rule implies that $\lim_{t\to\infty} e^{-(\rho-n)t}\theta(t)\Phi(t) = [\log\lambda/(\rho - n)]\,\iota(t)/\infty$, which equals zero for a finite level of $\iota(t)$ which follows from the constraint $(1-s)/a_\iota d(t) \ge \iota(t) \ge 0$.

Substituting for $\theta(t) = \log\lambda/(\rho - n)$ into (B2) gives:

$$\iota = \iota(\eta, d) = \frac{1}{d}\left[\frac{1-s}{a_\iota} - \frac{1}{\log\lambda/[d(\rho-n)] + \eta\mu}\right],$$

where $\partial\iota(\eta,d)/\partial\eta > 0$, $\partial\iota(\eta,d)/\partial d < 0$. Substituting $\iota(\eta,d)$ and $\theta(t)$ into the costate equation for η (B4) along with the equation of motion for d implies the following autonomous system:

$$\dot\eta = \eta\left[\rho - \mu\iota(\eta,d)\right] + \frac{a_\iota}{\left[(1-s)/\iota(\eta,d)\right] - a_\iota d},$$

$$\dot d = \frac{\delta s}{\gamma} + [\mu\iota(\eta,d) - n]d.$$

At the steady-state for $\tilde{d} > 0$, we need $n - \iota\mu > 0$. So, we restrict attention the neighborhood of the steady-state where $\iota < n/\mu$ holds. Observe that $d\dot{\eta}/d\eta$ and thus $(dd/d\eta)|_{\dot{\eta}=0}$ cannot be signed unambiguously. On the other hand, around the steady-state $d\dot{d}/dd < 0$ and $d\dot{d}/d\eta > 0$; thus, $(dd/d\eta)|_{\dot{d}=0} > 0$. In other words, $\dot{d} = 0$ is upward sloping, and starting from $\dot{d} = 0$ an increase (decrease) in d renders $\dot{d} < 0$, causing d to decrease (increase). Given that a unique steady-state equilibrium exists, we can examine all of the possibilities. First, when $\dot{\eta} = 0$ is upward sloping, $\dot{\eta} = 0$ may be steeper of flatter than $\dot{d} = 0$ and $d\dot{\eta}/d\eta$ maybe positive or negative. Hence, with $\dot{\eta} = 0$ upward sloping, there are four possibilities. It can be shown graphically that the system is saddle path stable in either case. Second, when $\dot{\eta} = 0$ is downward sloping, $d\dot{\eta}/d\eta$ may be positive or negative. Thus, with $\dot{\eta} = 0$ there are two possibilities. It can be shown graphically that the system is saddle-path stable when $d\dot{\eta}/d\eta > 0$ and exhibits a stable focus when $d\dot{\eta}/d\eta < 0$.

Appendix C: Identifying the Marginal Impact of Innovation

I now derive an expression for the welfare impact of a marginal innovation, following closely Grossman and Helpman (1991, pp.110-111) and Segerstrom (1998, pp. 1308-1309). I consider a situation in which an external agent becomes successful in innovating a higher quality product in industry ω at time $t = 0$. I then investigate the impact of this event on the welfare of all economic agents other than the external agent. To do this, I perturb the competitive equilibrium solution by $d\Phi$ at time $t = 0$ and investigate the impact on the discounted welfare for the period $(0, \infty)$. I exclude the welfare of the external agent from the analysis because the free-entry in R&D condition implies that for any entrepreneur engaged in R&D, the R&D costs must be exactly balanced by the expected discounted rewards from R&D. Note that with the measure one of structurally-identical industries, it again follows that $D(\omega,t) = D(t)$, $\iota(\omega,t) = \iota(t)$, $X(\omega,t) = X(t)$ for all ω and t.

Let $E(t) = ce^{nt}$ denote the aggregate consumer expenditure. Thus, (30) can be restated as:

$$U = \int_0^\infty e^{-(\rho-n)t}\left[\Phi(t)\log\lambda + \log\frac{E(t)}{e^{nt}\lambda}\right]dt$$

To derive the welfare impact of an incremental innovation, I differentiate the above:

$$\frac{dU}{d\Phi} = \int_0^\infty e^{-(\rho-n)t}\log\lambda dt + \int_0^\infty e^{-(\rho-n)t}\frac{1}{E(t)}\frac{dE(t)}{d\Phi(t)}dt\ .$$

The first term equals $\log\lambda/(\rho - n)$ and capture the *consumer surplus externality*. The second term captures the *business stealing* and *intertemporal R&D spillover externalities*. The successful innovation by the external agent leads to the replacement of the incumbent firm in industry ω, resulting in a loss of stream of monopoly profits and lower incomes for its stockholders. In addition, the marginal innovation raises the difficulty of future research, resulting in more resources being diverted to innovation activities. Both of these effects lead to lower consumption expenditure, which are compounded by multiplier effects.

I now explicitly identify the business stealing and intertemporal R&D externalities. Note that aggregate expenditure equals aggregate income minus aggregate savings:

$$E(t) = [sw + (1 - s)]e^{nt} + \pi(t) - \iota a_\iota D(t),$$

where the first term measures the labor income from specialized and non-specialized labor, the second term measures the aggregate profit income (excluding that of the external agent), and the third term measures the aggregate investment in R&D. Differentiating $E(t)$ with respect to Φ and noting $d\Phi/d\iota = 0$ gives:

$$\frac{dE(t)}{d\Phi(t)} = se^{nt}\frac{dw}{d\Phi} + \frac{d\pi}{d\Phi} - \iota a_\iota \frac{dD(t)}{d\Phi} \quad .$$

Recall that by combining (13) and (19), which hold both in and out of the steady-state, we have derived $w = a_\iota\delta(1 - \phi_\iota)\iota/\gamma(1+\mu\iota)$ thus $dw/d\Phi = 0$. Next, I consider $d\pi/d\Phi$. This term captures the decline in aggregate expenditure associated with the loss of profits in industry ω due to the marginal innovation. In this model, the effective replacement rate takes into account the rent protection costs of incumbent firms and equals $\iota(1 + \eta(\iota))$. Since the arrival of innovations is governed by a Poisson process whose intensity equals ι, the arrival of effective replacement incidents is also governed by a Poisson process whose intensity equals $\iota(1 + \eta(\iota))$. It follows from the properties of the Poisson process that the effective duration of monopoly power is exponentially distributed with parameter $\iota(1 + \eta(\iota))$. In the event of no further innovation between time 0 and time t— that is, between the time the external agent innovates and the time that signifies the current period—with probability $e^{-\iota(1+\eta(\iota))t}$, no further innovation occurs and the economy forfeits monopoly profits at time t of amount $(\lambda - 1)E(t)/\lambda$. In addition there is a multiplier effect because the loss of profits in industry ω will induce a fall in aggregate expenditure, which will translate into lower incomes and expenditures in other industries. The combined change in total profits equals:

$$\frac{d\pi}{d\Phi} = -\frac{\lambda - 1}{\lambda}E(t)e^{-\iota(1+\eta(\iota))} + \frac{\lambda - 1}{\lambda}\frac{dE}{d\Phi} \, .$$

Next, consider $\iota a_\iota \dfrac{d(D(t))}{d\Phi}$, which captures the expenditure decline associated with the higher resource requirement in R&D due to marginal innovation. The first step is to find the solution to the differential equation (7). Note that $D(\omega,t) = D(t)$, $\iota(\omega,t) = \iota(t)$ and $X(\omega,t) = X(t)$ for all ω and t. In addition, $X_A(t)= X(t)$ and $\iota_A(t)= \iota(t)$. The solution to (7) is then given by:

$D(t) = D_0 e^{\Phi(t)\mu} + \frac{\delta s}{\gamma} e^{\Phi(t)\mu} \int_0^t e^{ns-\mu\Phi(s)} ds$, where D_0 is the level of R&D difficulty at time 0.

Differentiating this gives:

$$\frac{dD(t)}{d\Phi} = \mu D_0 e^{\mu\Phi(t)} .$$

Substituting the expressions for $d\pi/d\Phi$ and $dD(t)/d\Phi$ into the $dE(t)/d\Phi$ expression and collecting terms gives:

$$\frac{dE(t)}{d\Phi(t)} \frac{1}{E(t)} = -(\lambda - 1) e^{-\iota[1+\eta(\iota)]} - \frac{1}{E} \lambda \iota a_\iota \mu D_0 e^{\mu\Phi(t)} .$$

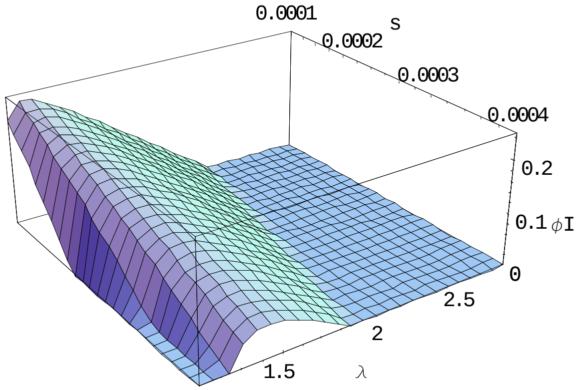
Substituting the above expression now into $dU/d\Phi$ using $E = [\lambda a_\iota D(t)(\rho + \iota(1 + \eta(\iota)) - n)]/(\lambda - 1)$, which follows from (25) and (13), and calculating the integral implies:

$$\frac{dU}{d\Phi} = \frac{\log\lambda}{\rho - n} - \frac{\lambda - 1}{\rho + \iota\left[1 + \eta(\iota)\right] - n} - \frac{\mu\iota(\lambda - 1)}{\rho + \iota\left[1 + \eta(\iota)\right] - n} \int_0^\infty \frac{D_0 \mathrm{e}^{-(\rho-n)t+\mu\Phi(t)}}{D(t)} dt$$

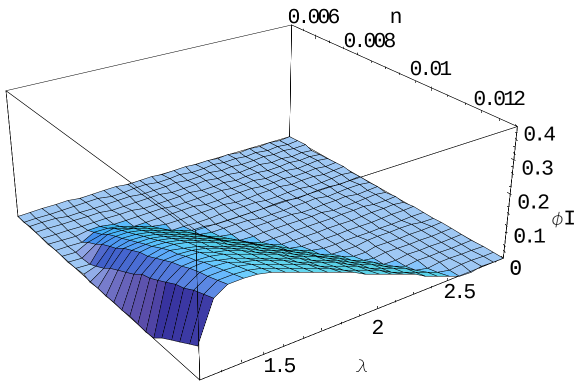
To evaluate the integral term, I invoke the steady-state properties of the model as in Segerstrom (1998, p. 1309). At the steady-state $\iota(t)= \iota$, $\Phi(t)= \iota t$ and $D(t) = D_0 e^{nt}$. Substituting these into the integral term and evaluating gives (33).

Appendix D: 3D Simulations, λ vs ϕ_ι and a generic parameter $\alpha \in \{s, n, A_\iota, \mu, \rho\}$

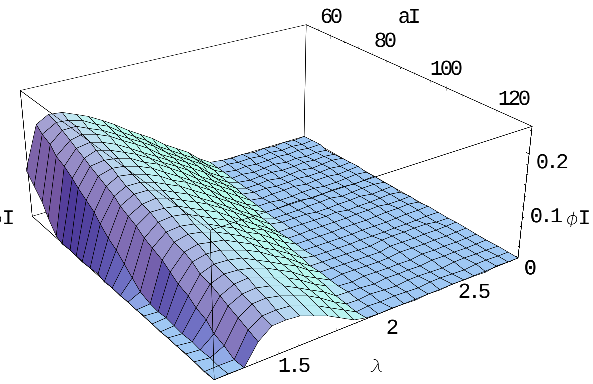
The case of s



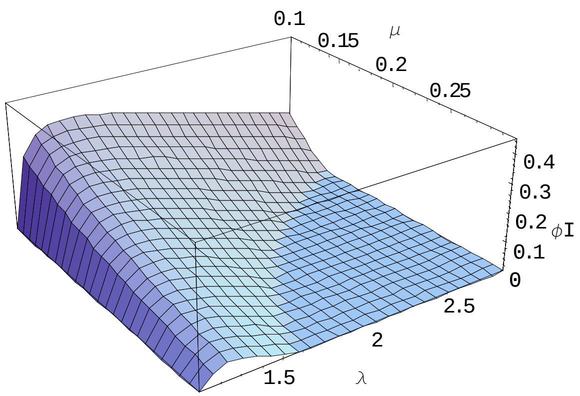
The case of n



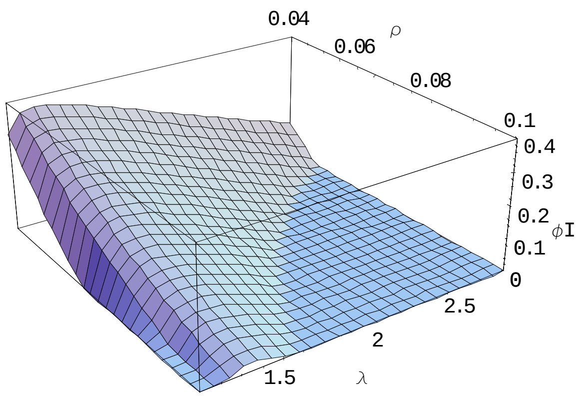
The case of A_ι



The case of μ



The case of ρ



Appendix E: The Endogenous *s* Model

I now reconstruct the model by assuming that labor is of one type and fully mobile across manufacturing, R&D and RPAs. I study the steady-state and welfare properties of the endogenous *s* model and compare them against the baseline model.

I first normalize the wage rate of labor to one. All of the equations of the model remain the same with the following exceptions. In equations (9), (18) and (19), the *w* terms must be set to one. For future use, I rewrite the monopolist's profit flows, (9), and the optimality condition for RPAs, (19), with ω terms dropped out due to structural symmetry:

$$\pi(t) = \frac{\lambda - 1}{\lambda} c(t) N(t) - \gamma X(t), \tag{9$'$}$$

$$\frac{\delta \iota(t) \upsilon(t)}{D(t)[1 + \mu \iota(t)]} = \gamma \ . \tag{19$'$}$$

With full labor mobility, the labor market equilibrium condition becomes:

$$N(t) = R_A(t) a_\iota + \frac{c(t)N(t)}{\lambda} + \gamma X_A(t) \ , \tag{20$'$}$$

where the demand for labor now comes from R&D, manufacturing, and also RPAs.

E.1. Steady-State Equilibrium

At the steady state $\iota(t)$, $c(t)$ and $r(t)$ remain constant over time, and $X(t)$, $\upsilon(t)$, $D(t)$ and $\pi(t)$ grow at the rate of n. I henceforth drop the time index for the variables that remain constant. Equations (22) through (25) remain the same with the exception of (24), which now becomes:

$$\gamma X(t) = \iota \eta(t) \upsilon(t). \tag{24'}$$

Combining (19′) with (13) gives the "Relative Profitability" Condition:

$$\frac{\frac{\upsilon(t)}{D(t)}}{\frac{\delta \iota}{(1+\mu \iota)} \frac{\upsilon(t)}{D(t)}} = \frac{a_\iota (1 - \phi_\iota)}{\gamma} \quad \Rightarrow \quad \frac{1}{\frac{\delta \iota}{1 + \mu \iota}} = \frac{a_\iota (1 - \phi_\iota)}{\gamma}, \qquad \textbf{RP}$$

where the LHS is the rewards from R&D relative to the rewards from RPAs, whereas the RHS is the cost of innovation relative to the cost of RPAs. Hence, any parameter change that increases the relative profitability of innovation with respect to RPAs leads to a rise in ι. [27] Indeed one can find a closed form solution for the Competitive Equilibrium (CE) level of ι:

[27] The LHS is decreasing in ι, because a higher ι increases the threat of replacement rendering R&D less profitable relative to RPA.

$$\iota^* = \frac{1}{A_\iota(1-\phi_\iota)-\mu},$$

where $A_\iota = a_\iota\delta/\gamma$. This implies that growth is endogenous in the sense that R&D policies that work through ϕ_ι can affect ι^*. However, ι^* is not a function of all of the model's parameters; hence, in that sense growth is *not* fully endogenous. In particular ι^* is unaffected by changes in ρ, n and λ. The reason is that with only one type of labor, the relative profitability of R&D relative to RPA is no longer tied to a relative wage (see RP condition). The absence of such a link eliminates the feedback effects of ρ, n and λ on ι^*. Adding one more type of labor in either R&D/RPA or both reestablishes the link between relative profitability and relative wage, linking ι^* to all of the model's parameters.

Substituting $\upsilon(t)$ from (25), $\eta(\iota)$ from (23), $D(t)$ from (22) and ι^* into the free-entry in R&D condition (13) gives

$$\frac{\delta x}{[n-\mu\iota^*]}a_\iota(1-\phi_\iota) = \frac{[(\lambda-1)/\lambda]c}{\rho-n+\iota^*\left[1+\frac{(n-\mu\iota^*)}{(1+\mu\iota^*)}\right]}, \qquad \textbf{RD}^{\textbf{CE}\prime}$$

where $x = X(t)/N(t)$. Substituting for $D(t)$ from (22) into the labor market condition (20') using $R(t) = \iota D(t)$ and structural symmetry across industries gives:

$$1 = \frac{c}{\lambda} + x\left(\frac{\delta a_\iota \iota^*}{n-\mu\iota^*}+\gamma\right). \qquad \textbf{LM}'$$

Equations RD^{CE} and LM constitute a system of two equations in two unknowns (x,c) and complete the solution of the steady-state equilibrium. It is straightforward to show that in (x,c) space, $\text{RD}^{\text{CE}\prime}$ is linear, upward sloping and starts from the origin; and that LM′ is linear, downward sloping and starts from a positive vertical intercept λ. The only parametric restriction for existence and uniqueness for strictly non-negative values for ι^*, c^* and x^* is that $\frac{n}{\mu} > \frac{1}{A_\iota(1-\phi_\iota)-\mu} > 0$.

E.2. Welfare Analysis

I now consider the social planner whose objective is to maximize welfare by choosing ι and taking the level of x as given. The optimal level of x is determined by the rate of return maximization of monopolist and thus follows from (24′). The planner has at her disposal only R&D subsidies/taxes but no policy that applies for RPAs. [28] In particular, the planner's objective is to

[28] One can instead assume that the social planner can choose both x and ι by using tax/subsidy tools for both R&D and RPAs. It is easy to show that the social planner then taxes RPAs prohibitively high and thus no labor performs RPAs. The model then effectively boils down to that of Segerstrom (1998) as far as welfare

$$\max_{\{\iota\}} \int_0^\infty e^{-(\rho-n)t} \left\{ \Phi(t) \log \lambda + \log[c/\lambda] \right\} dt \tag{E1}$$

subject to the state equations $\dot{\Phi} = \iota(t)$, $\dot{d} = \delta x + [\mu\iota(t) - n)]d(t)$, the labor market equilibrium condition $(c/\lambda) + a_\iota d\iota + \gamma x = 1$, initial conditions $\Phi(0) = 0$, $d(0) = d_0 > 0$, and the control constraint, $(1 - \gamma x) / a_\iota d(t) \geq \iota(t) \geq 0$ for all t. Substituting for c/λ, the current value Hamiltonian for this problem can be stated as:

$$H = \Phi \log\lambda + \log[1 - a_\iota d\iota - \gamma x] + \theta\iota + \eta[\delta x + (\mu\iota - n)d].$$

For an interior solution, the foc for ι is:

$$\frac{\partial H}{\partial \iota} = -\frac{a_\iota d}{1 - a_\iota d\iota - \gamma x} + \theta + \eta\mu d = 0 \tag{E2}$$

The costate equations are

$$\dot{\theta} = (\rho - n)\theta - \frac{\partial H}{\partial \Phi} = (\rho - n)\theta - \log\lambda \tag{E3}$$

$$\dot{\eta} = (\rho - n)\eta - \frac{\partial H}{\partial d} = (\rho - n)\eta + \frac{a_\iota \iota}{1 - a_\iota d\iota - \gamma x} - \eta(\mu\iota - n) \tag{E4}$$

I solve the optimal control problem for a BGP in which all endogenous variables grow at constant rates (although not necessarily the same rate) and $\iota > 0$. Using arguments analogous to those in Appendix B, it follows that ι, x, c, d, θ, and η must remain constant at the BGP.

Imposing $\dot{\theta} = 0$ on (E3) implies $\theta^* = \log\lambda/(\rho - n)$. Imposing $\dot{\eta} = 0$ on (E4) and using $c/\lambda = 1 - a_\iota d\iota - \gamma x$ gives $\eta(\iota,c,x) = a_\iota\iota\lambda/ c(\iota\mu - \rho)$. Imposing $\dot{d} = 0$ gives $d(x,\iota) = \dfrac{\delta x}{(n - \iota\mu)}$. Substituting for θ^*, $\eta(\iota,c,x)$ and $d(x,\iota)$ in (E2) yields the Socially Optimal (henceforth **SO**) foc for ι in terms of (ι,c,x):

$$\frac{c(\log\lambda)}{\lambda(\rho - n)} = \frac{a_\iota \delta x \rho}{(n - \mu\iota)(\rho - \mu\iota)} \qquad \mathbf{RD^{SO\prime}}$$

Combining (24′) and (25), using $c/\lambda = 1 - a_\iota d\iota - \gamma x$ gives the foc for x in terms of (ι,c,x):

$$\gamma x = \frac{\iota(n - \iota\mu)[(\lambda - 1)/\lambda]c}{(1 + \mu\iota)\left\{\rho - n + \iota\left(1 + \dfrac{n - \mu\iota}{1 + \mu\iota}\right)\right\}} \tag{E5}$$

Combining (E5) and $\mathrm{RD}^{SO\prime}$ gives the SO Relative Profitability condition between R&D and RPA:

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implications are concerned. Hence, I choose to investigate the more interesting case by forcing the planner to take x as given. Thanks are to a referee for suggesting this route.

$$\frac{\dfrac{\dfrac{\log\lambda}{\rho-n}}{\lambda-1}}{\rho-n+\iota\left(1+\dfrac{n-\mu\iota}{1+\mu\iota}\right)}\frac{1}{\dfrac{\delta\iota}{1+\mu\iota}}=\frac{a_\iota}{\gamma}\frac{\rho}{\rho-\mu\iota} \qquad \mathbf{RP^{SO\prime}}$$

Finally, using the expression for *d(x,ι)* gives the labor market condition in *(ι,c,x)*:

$$1=\frac{c}{\lambda}+x\left(\frac{\iota\delta a_\iota}{n-\iota\mu}+\gamma\right) \qquad \mathbf{LM\prime}$$

Equations $RD^{SO\prime}$, $RP^{SO\prime}$ and LM′ constitute a system of three equations which determine the SO vector *(ι,c,x)*. These equations are basically the SO counterparts of RD^{CE}, RP^{CE} and LM. Observe that RP^{SO} implicitly determines the welfare-maximizing level of *ι*. The labor market condition obviously remains the same for both problems as expected.

E.3. Simulations

To determine the optimal R&D policy, I run numerical simulations. I choose the following benchmark parameters: $\lambda = 1.25$, $\rho = 0.07$, $n = 0.01$, $a_\iota = 45$, $\mu = 0.20$, $\delta = 1$, and $\gamma = 1$. These are the same as the baseline values with two exceptions: parameter *s* is omitted because there is only one type of labor, and a_ι is set at 45 to generate a growth rate *g* around 0.5 percent.

The benchmark simulation implies that the competitive equilibrium innovation rate is $\iota^{CE} =$ 0.0223214. Let s_M, s_{RD}, and s_X represent the employment share of manufacturing, R&D, and RPA workers, respectively. The competitive equilibrium levels are $s_M^{CE} = 0.9362776$, $s_R^{CE} = 0.0633731$ and $s_X^{CE} = 0.0003492$. The socially optimal levels are $\iota^{SO} = 0.0266409$, $s_M^{SO} = 0.91584$, $s_R^{SO} = 0.0838336$, $s_X^{SO} = 0.0003266$. Clearly, the $\iota^{SO} > \iota^{CE}$; hence, the optimal policy is a subsidy which is calculated as $\phi_\iota^{SO} = 0.161415$.

In this benchmark case, the competitive markets underinvest in R&D and overinvest in manufacturing and RPAs. The socially optimal outcome requires that resources be drawn from *both* manufacturing and RPA to increase the employment share of R&D workers. When ϕ_ι^{SO} is mapped against λ, a downward sloping curve emerges as shown in Figure 4. This is because ι^{CE} is independent of λ, and thus the relationship between the welfare effects of a marginal innovation MU_Φ and λ can now be signed unambiguously as $dMU_\Phi/d\lambda < 0$. This is of course different from ambiguity found in the baseline model and implied n-shaped curve in $(\lambda, \phi_\iota^{SO})$ in that was found in the baseline model. Not surprisingly, however, similar to Segerstrom’s (1998) finding, where $\iota^{CE} = n/\mu$, and ϕ_ι^{SO} is decreasing in λ.

Appendix F: Generalizing the Production Function for RPAs

I now assume that RPA uses both general-purpose and specialized labor. As before, manufacturing and R&D uses only general-purpose labor. The population shares of specialized and general-purpose workers are given by s and $(1-s)$. I consider a cost function derived from a standard Cobb-Douglass production function with constant returns to scale. The unit cost of conducting RPAs is:

$$B = \gamma w_S^{\beta} w_G^{1-\beta},$$

where $1 \geq \beta \geq 0$, $\gamma > 0$, w_S and w_G stand for the wage rates of specialized and general-purpose labor, respectively.[29] Using Shephard's Lemma, the RPA unit labor requirements for specialized and general-purpose workers can be expressed as:

$$b_S = \frac{\partial B}{\partial w_S} = \gamma \beta w^{\beta-1} ,$$

$$b_G = \frac{\partial B}{\partial w_G} = \gamma (1-\beta) w^{\beta} ,$$

where $w = w_G/w_S$. Normalizing w_G to one, the unit cost of RPA simplifies to $B = \gamma w^{\beta}$.

All of the equations of the baseline model remain the same except for the following. Using the unit cost $B = \gamma w^{\beta}$, I restate the monopolist's profit flows (9) as:

$$\pi(t) = \frac{\lambda - 1}{\lambda} c(t) N(t) - \gamma w^{\beta} X(t) . \tag{9''}$$

Using again $B = \gamma w^{\beta}$, I rewrite the optimality condition for RPAs (19) as:

$$\frac{\delta \iota(t) \upsilon(t)}{D(t)[1+\mu \iota(t)]} = \gamma w^{\beta} . \tag{19''}$$

With $b_G = \gamma(1-\beta)w^{\beta}$, the labor market equilibrium condition for general-purpose labor becomes:

$$N(t) = R_A(t) a_{\iota} + \frac{c(t)N(t)}{\lambda} + \gamma(1-\beta) w^{\beta} X_A(t) . \tag{20''}$$

With $b_S = \gamma \beta w^{\beta-1}$, the labor market equilibrium condition for specialized labor becomes:

$$sN(t) = X_A \gamma \beta w^{\beta-1} . \tag{21''}$$

F.1. Steady-State Equilibrium

At the steady state $w(t)$, $\iota(t)$, $c(t)$ and $r(t)$ remain constant, and $X(t)$, $\upsilon(t)$, $D(t)$ and $\pi(t)$ grow at the rate of n. Equations (22) through (25) remain the same with the exception of (24), which now becomes:

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[29] Obviously, with $\beta = 1$, it follows that $B = \gamma w_S$ and the model collapses to the baseline setting. With $\beta = 0$, it follows that $B = \gamma w_G$ and the model collapses to the homogeneous labor case of Appendix E.

$$\gamma w^{\beta} X(t) \;=\; \iota\eta(\iota)\upsilon(t). \qquad (24'').$$

Substituting for $X_A(t)$ from (21″) into (22) using $X_A(t) = X(t)$ and solving for $D(t)$ gives

$D(w,\iota,t) = \dfrac{\delta s N(t) w^{1-\beta}}{\gamma\beta(n-\iota\mu)}$. Substituting this and $\upsilon(t)$ from (25) into the R&D condition (13) gives:

$$\frac{A_\iota s(1-\phi_\iota)w^{1-\beta}}{\beta(n-\mu\iota)} = \frac{[(\lambda-1)/\lambda]c}{\rho - n + \iota\left[1+\dfrac{(n-\mu\iota)}{(1+\mu\iota)}\right]}, \qquad \mathbf{RD^{CE''}}$$

where $A_\iota = a_\iota/(\delta\gamma)$. Combining (13) and (19″) and solving for w gives the Relative Wage (RW) equation:

$$w = \left[\frac{A_\iota \iota(1-\phi_\iota)}{1+\mu\iota}\right]^{\frac{1}{\beta}}. \qquad \mathbf{RW}$$

Substituting $D(w,\iota,t)$ and X_A from (21″) into (20″), using $R_A(t) = R(t) = \iota D(w,\iota,t)$ implies:

$$1-s = \frac{A_\iota s\iota}{\beta(n-\mu\iota)} w^{1-\beta} + \frac{c}{\lambda} + \frac{s(1-\beta)}{\beta} w. \qquad \mathbf{LM''}$$

The equations RD^{CE}, RW and LM″ constitute a system of 3 equations in 3 unknowns (c, ι, w). Solving for c/λ from $RD^{CE''}$ and substituting it into LM" implies:

$$1-s = \frac{A_\iota s w^{1-\beta}}{\beta(n-\mu\iota)}\left[\iota + \frac{(1-\phi_\iota)}{\lambda-1}\left(\rho - n + \frac{1+n}{\mu + (1/\iota)}\right)\right] + \frac{s(1-\beta)}{\beta} w. \qquad \mathbf{LM'''}$$

Hence, now LM‴ and RW can be solved for w and ι. It is easy to show that in (ι,w) space, RW is upward sloping and LM‴ is downward sloping. There exists a unique equilibrium level in the range $\iota < n/\mu$, without requiring further restrictions. [30]

F.2. Welfare Analysis

I now consider the social planner whose objective is to maximize welfare by choosing ι taking the level of x as given (as in Appendix E). In particular, the planner's objective is to

$$\max_{\{\iota\}} \int_0^{\infty} e^{-(\rho-n)t}\left\{\Phi(t)\log\lambda + \log[c/\lambda]\right\} dt \qquad \text{(F1)}$$

subject to the state equations $\dot{\Phi} = \iota(t)$, $\dot{d} = \delta x + [\mu\iota(t) - n)]d(t)$, the general-purpose labor market equilibrium condition $(c(t)/\lambda) + a_\iota d\iota(t) + \gamma x(t)\,(1-\beta)w^{\beta}(t) = 1 - s$, the specialized labor market

[30] On the LM curve when $\iota \to 0$, $w \to w^1 > 0$, where w^1 is the solution to $[(1-s)n\beta]/s = \{[A_\iota w^{1-\beta}(1-\phi_\iota)(\rho-n)]/(\lambda-1)\} + [(1-\beta)w]$; and when $\iota \to \iota^{max} = n/\mu$, $w \to 0$. On the RW curve, when $\iota \to \iota^{max}$, $w \to w^2 = \{[A_\iota(1-\phi_\iota)n]/[\mu(1+n)]\}^{1/\beta}$ and when $\iota \to 0$, $w \to 0$.

equilibrium condition $x(t) = (sw(t)^{1-\beta})/(\gamma\beta)$, initial conditions $\Phi(0) = 0$, $d(0) = d_0 > 0$, and the control constraint, $[1 - s - x(t)\gamma(1-\beta)\gamma w(t)^{\beta}]/(a\iota d(t)) \geq \iota(t) \geq 0$ for all t. The optimal level of $x(t)$ follows from (24”). Substituting for $c(t)/\lambda$ and $x(t)$ from the general-purpose and specialized labor market conditions, the current value Hamiltonian for this problem can be stated as:

$$H = \Phi \log\lambda + \log\left[1 - s - a_\iota d\iota - \frac{s(1-\beta)}{\beta}w\right] + \theta\iota + \eta\left[\frac{\delta s}{\gamma\beta}w^{1-\beta} + (\mu\iota - n)d\right]$$

For an interior solution, the foc for ι is:

$$\frac{\partial H}{\partial \iota} = -\frac{a_\iota d}{1 - s - a_\iota d\iota - \frac{s(1-\beta)}{\beta}w} + \theta + \eta\mu d = 0 \qquad \text{(F2)}$$

The costate equations are

$$\dot{\theta} = (\rho - n)\theta - \frac{\partial H}{\partial \Phi} = (\rho - n)\theta - \log\lambda = 0 \qquad \text{(F3)}$$

$$\dot{\eta} = (\rho - n)\eta - \frac{\partial H}{\partial d} = 0 = (\rho - n)\eta + \frac{a_\iota \iota}{1 - s - a_\iota d\iota - \frac{s(1-\beta)}{\beta}w} - \eta(\mu\iota - n) \qquad \text{(F4)}$$

I solve the optimal control problem for a BGP in which all endogenous variables grow at constant rates (although not necessarily the same rate) and $\iota > 0$. Using arguments analogous to those in Appendix B, it follows that w, ι, x, c, d, θ, and η must remain constant at the BGP.

Imposing $\dot{\theta} = 0$ on (F3) implies $\theta^* = \log\lambda/(\rho - n)$. Imposing $\dot{\eta} = 0$ on (F4) and using $c/\lambda = 1 - s - a_\iota d\iota - \gamma x(1-\beta)w^{\beta}$ gives $\eta(\iota,c) = a_\iota\iota\lambda/c(\iota\mu - \rho)$. Imposing $\dot{d} = 0$ yields $d(w,\iota) = [\delta s w^{1-\beta}]/[\gamma\beta(n - \mu\iota)]$. Substituting for θ^*, $\eta(\iota,c)$, and $d(w,\iota)$ into (F2) yields the foc for ι in terms of (ι,c,w):

$$\frac{A_\iota \rho s}{\beta(n - \mu\iota)(\rho - \mu\iota)}w^{1-\beta} = \frac{c(\log\lambda)}{\lambda(\rho - n)} \qquad \textbf{RD}^{\textbf{SO}}$$

Combining (24”) and (25) using $x = (sw^{1-\beta})/(\gamma\beta)$ gives the Relative Wage (RW) equation in (ι,c,w):

$$\frac{sw}{\beta} = \iota\left(\frac{n - \mu\iota}{1 + \mu\iota}\right)\frac{c(\lambda - 1)}{\lambda\left[\rho - n + \iota\left(1 + \frac{n - \mu\iota}{1 + \mu\iota}\right)\right]}. \qquad \textbf{RW}$$

Substituting for $d(w,\iota)$ into the general-purpose labor market equilibrium condition gives in (ι,c,w):

$$1 - s = \frac{A_\iota s\iota}{\beta(n - \mu\iota)}w^{1-\beta} + \frac{c}{\lambda} + \frac{s(1-\beta)}{\beta}w. \qquad \textbf{LM}''$$

The equations RD^{SO}, RW and LM″ constitute a system of three equations which determine the SO vector *(ι,c,w)*. These equations are basically the SO counterparts of RD^{CE}, RW^{CE} and LM″. Observe that the labor market condition remain the same for both problems as expected.

F.3. Simulations

To determine the optimal R&D policy, I run numerical simulations. I use the same benchmark parameters in the baseline model with two exceptions. I set $\beta = 0.75$ as a reasonable value. I set A_ι = 45 to make the model comparable with the benchmark homogeneous labor case.

The benchmark simulation implies that the competitive equilibrium innovation rate is ι^{CE} = 0.0152659. Let s_M, s_{RD}, and s_X represent the employment share of manufacturing, R&D, and RPA workers in general-purpose workers, respectively. The competitive equilibrium levels are s_M^{CE} = 0.951507, s_R^{CE} = 0.0481799 and s_X^{CE} = 0.0000834197. The socially optimal levels are ι^{SO} = 0.0169587, s_M^{SO} = 0.943086, s_R^{SO} = 0.0565989, s_X^{SO} = 0.0000854187. Clearly, $\iota^{SO} > \iota^{CE}$; hence, the optimal policy is a subsidy which is calculated as ϕ_ι^{SO} = 0.0833807.

The benchmark simulation implies that the competitive markets underinvest in R&D and overinvest in manufacturing. With two types of labor employed in RPA, we also observe an underinvestment in RPAs. The socially optimal outcome requires that resources be drawn from manufacturing to *both* RPA and RPAs. When ϕ_ι^{SO} is mapped against λ, the n-shaped curve that was found in the baseline model reemerges. Figure Appendix F illustrates this relationship for different levels of β. Observe that as the share of specialized labor β converges to zero (i.e., as the economy converges to homogeneous labor case) the n-shaped curve converges to a downward sloping line as predicted.

Figure Appendix F: Optimal R&D Subsidy ϕ_ι vs λ
(Two Types of Labor in RPA)