Damping in a variable mass on a spring pendulum

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A model of damped oscillations of a variable mass on a spring pendulum, with the mass decreasing at a constant rate is presented. The model includes damping terms that are linear and quadratic in the velocity. The effect of the mass loss rate on the magnitude of the damping parameters is discussed. The model is compared to experimental data obtained with the aid of computer-based laboratory equipment. © 2005 American Association of Physics Teachers. [DOI: 10.1119/1.1979498]

I. INTRODUCTION

Inexpensive computer-based equipment was recently used to measure the damped oscillations of a spring pendulum whose mass decreases at a constant rate,1

\[
m(t) = m_0 - rt, \tag{1}
\]

where \(m_0\) is the initial mass of the pendulum and \(r\) is the mass loss rate. The damping due to energy loss is caused by the mass loss and air drag. In Ref. 1 the air drag was assumed to be proportional to the velocity of the motion,

\[
F_d = -bv, \tag{2}
\]

where \(b\) is the damping constant, and an approximate expression was obtained for the decay of the oscillation amplitude as a function of time:

\[
A(t) = A_0 \left(1 - \frac{r}{m_0} t\right)^{b/2r+1/4}. \tag{3}
\]

The ratio, \(A(t)/A_0\), is independent of the initial amplitude \(A_0\). Three characteristic decays of the oscillation amplitude were identified, depending on which damping mechanism dominates.1 When air resistance plays the dominant role (\(b \gg r\)), the envelope is an exponential; when \(b \sim r\), the envelope is linear; and when the energy loss due to the mass loss dominates (\(b \ll r\)), the envelope is convex.

For the linear air drag model to be valid, the amplitude of the oscillations should be much smaller than the characteristic dimension of the body and the Reynolds number \(\text{Re}\) should satisfy \(\text{Re} \leq 3000\).2 In most cases this approximation does not account for the observed behavior of real pendulum, and nonlinear damping terms are generally necessary.3,4

In this paper we describe the damped oscillations of a variable mass spring pendulum for \(\text{Re} > 10^4\) by including an additional damping term due to air drag that is quadratic in the velocity. The approximate solution for the decay of oscillation amplitude is compared to data obtained with the aid of computer-based equipment.

II. THE MODEL

Consider a container filled with sand and suspended from a coil spring with force constant \(k\). The container oscillates along the vertical axis \(z\) and its mass \(m(t)\) decreases as in Eq. (1). The law of momentum conservation for a variable mass system is

\[
\frac{dm}{dt} v = F + u \frac{dm}{dt}, \tag{4}
\]

where \(v = dz/dt\), \(z\) increases downward, and \(u(dm/dt)\) is the rate at which the momentum leaves the system. The net external force, \(F\), on the container is due to its weight \(F_g = m(t)g\), where \(g\) is the acceleration of gravity and the restoring force, which is proportional to the displacement of the spring from its initial position \(z=0\), \(F_k = -k(z + z_0)\), where \(z_0(t) = m(t)g/k\) is the equilibrium position of the spring. The air drag, \(F_d\), is assumed to be a sum of terms that are linear and quadratic in the velocity

\[
F_d = -b(dz/dt) - c|dz/dt|(dz/dt), \tag{5}
\]

where \(c\) is the quadratic damping parameter. If we assume that the velocity of the sand relative to the container is zero at the moment it leaves the container, then \(u = v = dz/dt\) and Eq. (4) becomes

\[
m(t) \frac{d^2z}{dt^2} + c \left| \frac{dz}{dt} \right| \frac{dz}{dt} + b \frac{dz}{dt} + kz = 0. \tag{6}
\]

Equation (6) is nonlinear and cannot be solved analytically. Our approximate solution is based on the assumption that the damping is small and its effect on the change in the amplitude \(A\) and the angular frequency \(\omega\) over a single oscillation is small compared to the amplitude \(A_0\) and the frequency \(\omega_0 = \sqrt{k/m_0}\) without damping. For this assumption to be valid, we must have \(b \ll \sqrt{km} \), \(c \ll m/A\), and \(r \ll \sqrt{km}\).

The first approximation to the solution of Eq. (6) can be written as

\[
z(t) = A(t) \cos(\omega t + \phi), \tag{7}
\]

where \(A(t)\) slowly decreases, and \(\omega(t) \approx \sqrt{k/m(t)}\) slowly increases due to the mass loss; \(\phi\) is the initial phase. The functional form of \(A(t)\) can be found from the energy loss equation for a variable mass system, which in our case \((u \approx v)\) can be derived4 by taking of the scalar product of \(v\) with both sides of Eq. (4). The result is

\[
\frac{dE}{dt} = v F_d - r v^2. \tag{8}
\]

The energy loss over a single cycle is obtained by integrating Eq. (8) over the period \(T = 2\pi/\omega\) from \(t\) to \(t + T\). With Eq. (5), we have
Because \( A(t) \) and \( \omega(t) \) vary slowly over a single oscillation, we have from Eq. (7)

\[
v = \frac{dz}{dt} = A(t)\omega(t)\sin(\omega t + \phi) = X(t)\sin(\omega t + \phi),
\]

where \( X(t) = \omega(t)A(t) \). We substitute Eq. (10) in Eq. (9) and obtain

\[
\frac{\Delta E}{T} = -\left( b + \frac{r}{2} \right) \left( \langle v^n \rangle - c \langle v^3 \rangle \right),
\]

where \( \langle v^n \rangle \) denotes the average over a period:

\[
\langle v^n \rangle = \frac{\int_{t}^{t+T} \sin(\omega t + \phi)^n dt}{T} = \frac{\sin(\omega t + \phi)^n d\phi}{\pi}.
\]

We changed variables, \( \phi = \omega t \), and integrated over a quarter instead of a complete period. We define \( I_n = \int_0^{\pi/2} \sin^n \phi d\phi \) such that \( I_2 = \pi/4 \), for \( n=2 \), and \( I_2 = 2/3 \), for \( n=3 \).

Because \( A(t) \) decreases slowly, the total energy \( E(t) \) over a single period is conserved and can be written as

\[
E(t) = \frac{m(t)}{2} [A(t)\omega(t)]^2 = \frac{m(t)}{2} [X(t)]^2,
\]

or

\[
\frac{dE(t)}{dt} = m(t)X(t) \frac{dX(t)}{dt} + \frac{1}{2} \frac{d}{dt} \left[ X(t) \right]^2 + \frac{d}{dt} \left[ \frac{m(t)}{2} \right] \left[ X(t) \right]^2.
\]

Because the energy loss over a period is small, we can use the approximation

\[
\frac{\Delta E}{T} \approx \frac{dE}{dt},
\]

and equate the right-hand sides of Eqs. (11) and (14) and obtain

\[
m(t) \frac{dX(t)}{dt} = -\left( b + \frac{r}{2} \right) X(t) - \frac{4c}{3\pi} \left[ X(t) \right]^2.
\]

We then integrate Eq. (16) with initial condition \( X(0) = A_0 \omega_0 \) to give

\[
X(t) = \frac{A_0 \omega_0 (1 - rt/m_0)^{\alpha}}{1 + \beta [1 - (1 - rt/m_0)^{\alpha}]},
\]

where \( \alpha \) and \( \beta \) are

\[
\alpha = \frac{b}{2r} - \frac{1}{4},
\]

\[
\beta = \frac{16A_0 \omega_0 c}{3\pi(2b - r)}.
\]

Because \( X(t) = \omega(t)A(t) = A(t)\omega_0 (1 - rt/m_0)^{-1/2} \) where \( \omega_0 = \sqrt{k/m_0} \), Eq. (17) becomes

\[
\frac{\Delta E}{T} = -A_0 \omega_0 (1 - rt/m_0)^{-1/2} \frac{d}{dt} \left[ \frac{A(t)}{1 + \beta [1 - (1 - rt/m_0)^{\alpha}] \left[ A(t) \right]^2} \right],
\]

which expresses the decay of the amplitude over long-time scales compared to the period of oscillation. Unlike the linear damping case, the ratio \( A(t)/A_0 \) in Eq. (20) depends on the initial amplitude \( A_0 \) and the angular frequency \( \omega_0 \).

In the limit \( r \rightarrow 0 \), Eq. (20) yields the damping equation for a fixed mass pendulum

\[
A(t) = \frac{A_0 \exp(-\alpha rt)}{1 + \beta_0 (1 - \exp(-\alpha_0 rt))} (r \rightarrow 0),
\]

with \( \alpha_0 = b/2m_0 \), and \( \beta_0 = 8A_0 \omega_0 c/3\pi b \) (the subscript zero means that damping parameters are defined for \( r=0 \)). In the limit \( r \rightarrow \infty \), Eq. (20) yields

\[
A(t) = A_0 (1 - rt/m_0)^{1/4},
\]

that is, for a large mass discharge rate, the damping is governed by the mass loss. In this case \( A(t)/A_0 \) does not depend on \( A_0 \) and \( \omega_0 \) and has a convex envelope.

We will fit the experimental data to Eq. (20) by varying \( \alpha \) and \( \beta \) for a given value \( r \) and interpret the results in terms of the damping parameters \( b \) and \( c \).

III. EXPERIMENT

The experimental setup shown in Fig. 1 is the same as described in Ref. 1 with slight modifications. A coil spring with force constant \( k=30 \) N/m and unloaded length \( \sim 20 \) cm was connected vertically to a 50 N force sensor clamped on a metallic spindle rigidly anchored to the laboratory wall. The sensor has a resolution of 0.05 N and was connected to a Data Logger to a personal computer. The spring mass \((\sim 0.05 \text{ kg})\) was much less than the container mass \((\sim 1.5 \text{ kg})\) and can be neglected. The container was an inverted plastic soda bottle filled with sand and hooked from the bottom to the lower end of the spring. A thin wooden disk, 0.2 m in diameter and 4 mm thick, was mounted on the bottle neck to enhance the air drag. Specially designed aluminum funnel-shaped tips of different openings were used to vary the mass discharge rate. In all cases the mass discharge rate was constant and proportional to the outlet diameter to the \( \frac{5}{7} \) power.

The bottle was set to oscillate vertically by pulling it down for about 0.2 m from its equilibrium position and gently released, simultaneously allowing the sand to pour out. The force sensor logged the instantaneous force versus time at a
frequency of 10 s⁻¹. The force data (see Fig. 2) was imported to a spreadsheet for further analysis. The mass discharge rate \( r \) was determined from the slope of the line of \( F_t/m_0 \) and by directly measuring the weight loss rate at rest with the aid of the force sensor. The two methods give a slight difference. The \( F_t/m_0 \) data were converted to the equivalent displacement \( z_t = F_t/k \) about the stationary position, \( z_0 \), with standard error 2% and plotted as \( z_t/A_0 \) for various values of \( r \) (see Fig. 3). The peaks of the oscillations give \( A_t/A_0 \) which is fitted to Eq. (21) by varying \( \alpha \) and \( \beta \) (see Fig. 4). Typically, we used \( k = 30 \text{ N m}^{-1} \), \( m_0 \approx 1.5 \text{ kg} \), \( A_0 \approx 0.2 \), so that \( b < 1 \text{ kg s}^{-1} \), \( r < 1 \text{ kg s}^{-1} \), and \( c < 1 \text{ kg m}^{-1} \).

IV. RESULTS AND DISCUSSION

We first analyze the damping caused by air drag. Figure 3(a) shows typical oscillations for \( r = 0 \). Although the decay appears to be exponential, a fit of \( A_t/A_0 \) to Eq. (21) with \( \alpha_0 = 0.0035 \pm 0.0001 \text{ s}^{-1} \) and \( \beta_0 = 4.59 \pm 0.22 \) is much better than a fit to an exponential [see Fig. 4(a)]. The damping parameters calculated from \( \alpha_0 \) and \( \beta_0 \) using Table I are \( b_0 = 0.0107 \pm 0.0004 \text{ kg s}^{-1} \) and \( c_0 = 0.072 \pm 0.005 \text{ kg m}^{-1} \). Note that the effect of the air drag is apparent in both the parameters \( b \) and \( c \). If only the linear drag term was important, then only \( b \) would be nonzero.

Figure 3(b) shows the experimental results for \( r \approx 3.62 \text{ g s}^{-1} \). Almost exponential decay is observed for short time, but for longer times, the exponential fit is not as good, especially for large amplitudes which decay more slowly.
As follows from Fig. 6, for \( c \rightarrow 0, \) that is, air drag is unimportant and energy loss due to the mass loss rate determined from fitting the data to Eq. (20) becomes

\[
A(t) = A_0(1 - rt/m_0)^{1/2}.
\]  

(23)

As follows from Fig. 6, for \( r \approx 24 \text{ g s}^{-1}, \) \( c \approx 37 \text{ g s}^{-1}, \) Eq. (20) becomes

\[
A(t) = A_0(1 - rt/m_0)\frac{1}{r(c)}.
\]  

(24)

That is, the amplitude decays linearly with time, similar to the behavior in Fig. 3(d). The reasons for this behavior cannot be explained in terms of air drag for a fixed mass. For oscillatory motion of a body whose mass decreases at a constant rate, the drag force may depend on the viscosity \( \eta \) and the density \( \rho \) of the fluid, the characteristic dimension of the body \( d, \) the initial frequency \( \omega_0, \) the initial amplitude \( A_0, \) and the mass loss rate \( r. \) Dimensional analysis shows that the drag coefficient is a function of three dimensionless parameters: the Stokes number \( \beta = \rho_0 \omega_0 d^2 / \eta, \) the Keulegan-Carpenter number \( KC = A/d \) (or the oscillatory Reynolds number \( Re_0 = \rho_0 \omega_0 d / \eta = \beta KC, \) the counterpart of steady motion), and the dimensionless mass loss rate \( \lambda = r / \rho_0 d^2, \) that is,

\[
\frac{F_d}{\rho A_0^2 \omega_0 d^2} = \phi(Re_0, \lambda).
\]  

(25)

A study of this dependence is a separate problem. Here, we present qualitative trends, observed at \( Re_0 \approx 10^4: \)

- For \( \lambda \approx 0.5, \) both the linear and quadratic damping mechanisms are operative with \( b \) increasing and \( c \) decreasing with \( r \) (see Fig. 5). In this case, the time dependence of the envelope is concave as described by Eq. (20).

![Fig. 5. Damping parameters as a function of the mass loss rate determined from fitting the data to Eq. (20) for different values of \( r. \) Diamond (●): quadratic damping parameter \( c. \) Circle (○): linear damping parameter \( b. \)](image)

![Fig. 6. The quantity \( A_0(w_0/c + b)(r) \) as a function of the mass loss rate.](image)
• For $0.5 \leq \lambda < 1$, the quadratic damping parameter vanishes and $b$ decreases with $r$; $\lambda = 0.5$ corresponds to linear decay in the amplitude as described by Eq. (24); for $\lambda > 0.5$ the envelope is convex, a special case of which is Eq. (23).

• For $\lambda \gg 1$, both $b$ and $c$ are negligible and the decay of $A(t)$ due to the mass loss alone is given by Eq. (22).

V. CONCLUSION

We have derived approximate results for damped harmonic motion with a variable mass. By modeling the damping force, students can find the corresponding form of the damped oscillations. For instance, the oscillatory behavior of a sliding variable-mass-spring system with a linear damping term due to air drag and a velocity-independent dry friction (Coulomb friction) could be analyzed. Another model that could be tested experimentally is a variable mass vibration system with a damping force proportional to $|\omega|^n$, for arbitrary $n$. The experimental verification of a model would provide students with experience in dimensional analysis, data collection, and data analysis.

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