HW 1 – Linear Systems and Fourier Transforms  
(10 problems for 100 pts)

1) A useful mathematical tool is to translate a function $f(x,y)$ by a distance $(x_0, y_0)$. Believe it or not, this can be achieved by convolution! Given two functions $A(x,y)$ and $B(x,y)$, the 2-D convolution is:

$$A(x, y) * B(x, y) = \int_{-\infty}^{+\infty} A(\zeta, \eta)B(x - \zeta, y - \eta)d\zeta d\eta$$

Perform the 2-D convolution integral and use the point impulse properties to prove the following:

a) $f(x, y) * \delta(x, y) = f(x, y)$

Note: This simply means that convolving a function $f(x, y)$ with a unit impulse at the origin results in the same function $f(x, y)$.

b) $f(x, y) * \delta(x - x_0, y - y_0) = f(x - x_0, y - y_0)$

Note: This means that a function $f(x, y)$ can be shifted by $(x_0, y_0)$ via convolution with a unit impulse located at $(x_0, y_0)$.

2) Consider a 1-D LSI imaging system with $h(x) = \frac{1}{\Delta} \text{rect}(x/\Delta)$. The object is a bar phantom with a bar width and separation “w” (see sketch). Assume the bar phantom amplitude is either 1 or 0.

a) Sketch the output of the imaging system along the x-axis (i.e. $g(x,0)$) for the case $w = 2\Delta$.

Note #1: Think graphically!

Note #2: It might be easier to first think about the image of a single bar, then extrapolate to multiple bars.

b) What is the modulation $m_g$ of the image?

Note: $m_g = (g_{\text{max}} - g_{\text{min}})/(g_{\text{max}} + g_{\text{min}})$ for a non-sinusoidal image pattern. This is equivalent to Eq. 3.1 in the textbook.

c) Repeat for the case $w = \Delta$.

d) The image modulation becomes zero below a certain value of $w$. Determine this value. Again, think about this problem in a graphical manner.

3) An LSI imaging system produces an image $g(x, y) = f(x, y) * h(x, y)$, where the “*” is a 2-D convolution. When the object is a line impulse $f(x, y) = \delta(x)$, show that the resulting image (the line spread function) is given by:

$$l(x) = \int_{-\infty}^{+\infty} h(x, \eta)d\eta$$

Hint: As you do the 2-D convolution integral, notice that $f(\zeta, \eta)$ does not depend on $\eta$. Since the integration over $\eta$ goes from $-\infty$ to $+\infty$, it doesn’t matter whether the integrand is $h(x, y-\eta)$ or $h(x, \eta)$. 
4) Consider the function: \( f(x) = \text{rect}\left(\frac{x-d}{w}\right) \).
   a) Sketch \( f(x) \) and label important features.
   b) Compute the 1-D Fourier transform of \( f(x) \).
       Hint: You should get \( F(u) = wsinc(wu)e^{-j2\pi du} \)
   c) Sketch \( |F(u)| \) when \( w = 2 \text{ mm} \) and \( d = 1 \text{ mm} \) over \(-2 \leq u < 2 \text{ mm}^{-1} \).

5) It is a useful skill to leverage standard tables and properties of Fourier transforms — it can save you lots of work! A good demonstration for 2-D functions is Example 2.8 in the textbook. This problem deals with 1-D functions. The relevant properties for 1-D Fourier transforms are the following:

<table>
<thead>
<tr>
<th>1-D Signa</th>
<th>1-D Fourier Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rect function: ( \text{rect}(x) )</td>
<td>( \text{sinc}(u) )</td>
</tr>
<tr>
<td>Impulse: ( \delta(x) )</td>
<td>1</td>
</tr>
<tr>
<td>Translation: ( f(x - x_0) )</td>
<td>( F(u)e^{-j2\pi ux_0} )</td>
</tr>
<tr>
<td>Scaling: ( f(ax) )</td>
<td>( \frac{1}{</td>
</tr>
<tr>
<td>Convolution: ( f(x) * g(x) )</td>
<td>( F(u) \cdot G(u) )</td>
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</table>

We can compute the 1-D Fourier transform of \( f(x) = \text{rect}\left(\frac{x-d}{w}\right) \) with very little effort by using the properties of 1-D Fourier transforms!
   a) The 1-D Fourier transform of \( \text{rect}(x) \) is \( \text{sinc}(u) \). Determine the 1-D Fourier transform of \( \text{rect}\left(\frac{x}{w}\right) \), and make clear which property you are exploiting.
   b) Next, determine the 1-D Fourier transform of \( \text{rect}\left(\frac{x-d}{w}\right) \). Make clear which property you used.

6) It is well known that the 2-D Fourier transform of \( f(x,y) = e^{-\pi(x^2+y^2)} \) is \( F(u,v) = e^{-\pi(u^2+v^2)} \).
   a) Using the scaling property of the 2-D Fourier transform (see Table 2.2 of textbook), determine the 2-D Fourier transform of \( h(x,y) = \frac{1}{\pi w^2}e^{-\left(\frac{x^2+y^2}{w^2}\right)} \)
       Hint: You should get \( F(u,v) = e^{-\pi^2 w^2(u^2+v^2)} \)
   b) Sketch \( h(x,0) \) and \( H(u,0) \) for \( w = 1 \text{ mm} \).
   c) Sketch \( h(x,0) \) and \( H(u,0) \) for \( w = 4 \text{ mm} \).
   d) What is the effect of “\( w \)” on the shape and amplitude of \( h(x,0) \) and \( H(u,0) \)?
A key metric of an imaging system is the degree of blurring along a particular direction. This blurring can be described in the spatial domain (via the line spread function) or the frequency domain (via the modulation transfer function). The purpose of this problem is to show that a 1-D Fourier transform relates the two quantities. In class, we defined MTF(u) and H(u,v) as:

\[ MTF(u) = \frac{|H(u,0)|}{|H(0,0)|} \]
\[ H(u, v) = \iint_{-\infty}^{+\infty} h(x, y) e^{-j2\pi(ux+vy)} dxdy \]

a) Show that \( H(u,0) \) is the 1-D Fourier transform of the line spread function \( l(x) \).

Hint: Use the definition of \( H(u,v) \) and the result from Problem 3.

b) Derive a similar expression for \( H(0,0) \).

c) Show that \( MTF(u) = |L(u)|/|L(0)| \), where \( L(u) \) is the 1-D Fourier transform of \( l(x) \).

8) Suppose you measure an edge spread function described by \( esf(x) = \frac{\pi}{2} + \arctan(x/w) \). This function is shown in the figure to the right.

a) Compute the line spread function \( l(x) \).

Hint: Use the handy relation: \( \frac{d}{dx} \arctan(x) = \frac{1}{1+x^2} \)

b) Derive an expression for \( MTF(u) = |L(u)|/|L(0)| \), where \( L(u) \) is the 1-D Fourier transform of \( l(x) \).

Hint #1: The 1-D Fourier transform of \( f(x) = \frac{1}{1+x^2} \) is \( F(u) = \pi e^{-2\pi|u|} \).

Hint #2: You should get \( MTF(u) = e^{-2\pi w|u|} \).

c) Sketch MTF(u). Label important features, particularly the value of “u” where MTF = 1/e.

9) Turn in your figures and calculations from MiniLab1a.

10) Turn in your figures and calculations from MiniLab1b.